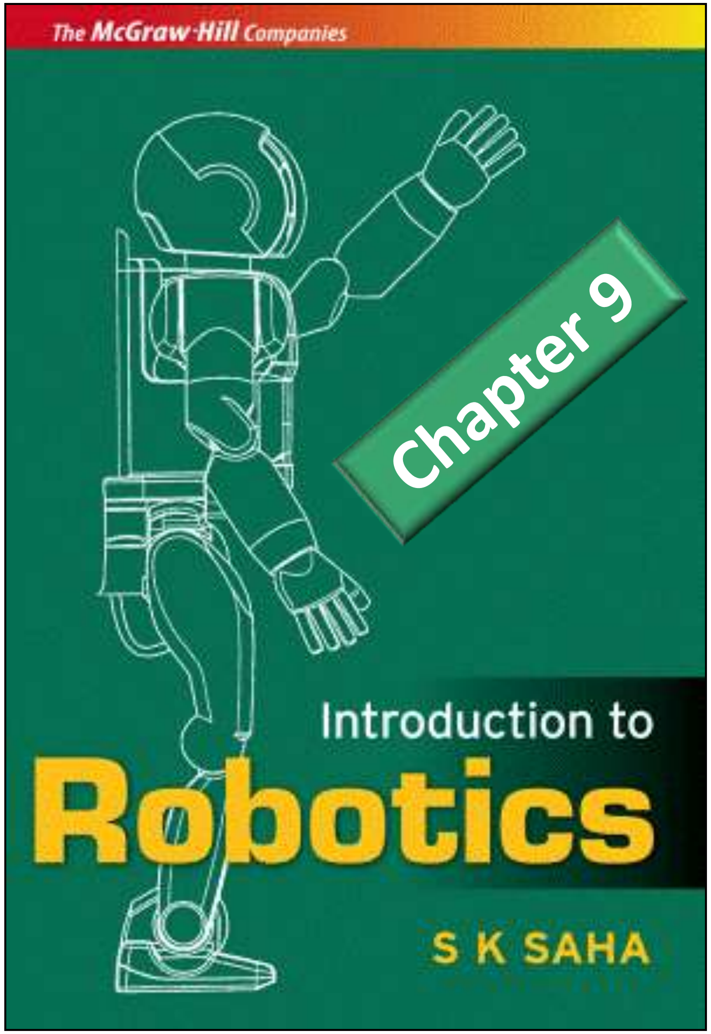

Lecture 1

Recursive Robot Dynamics

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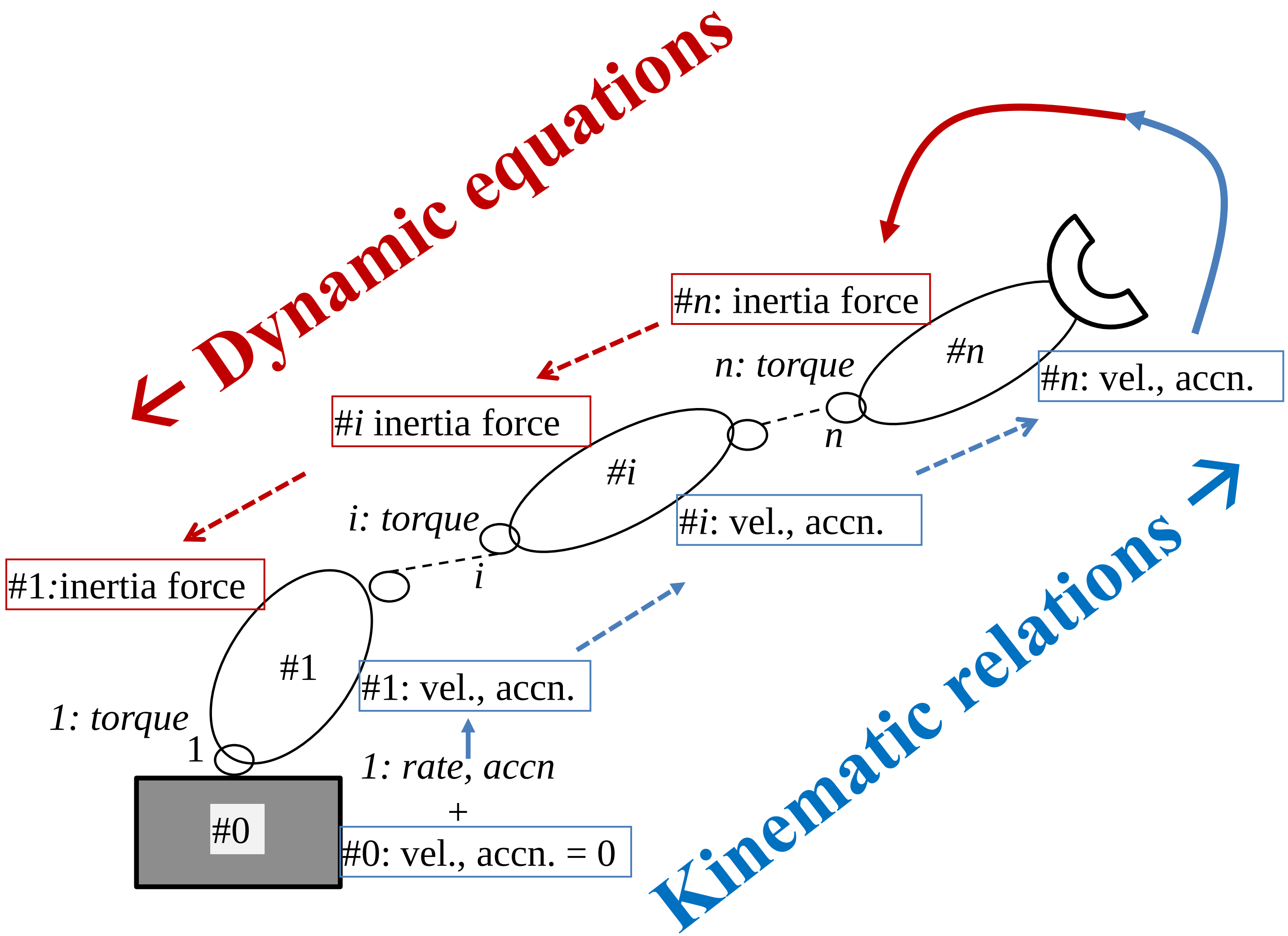
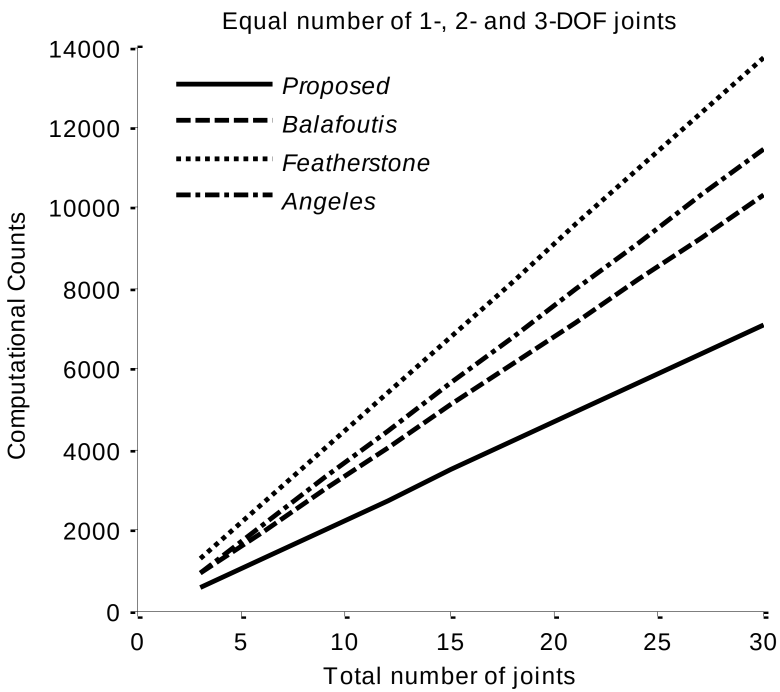
What is recursive?



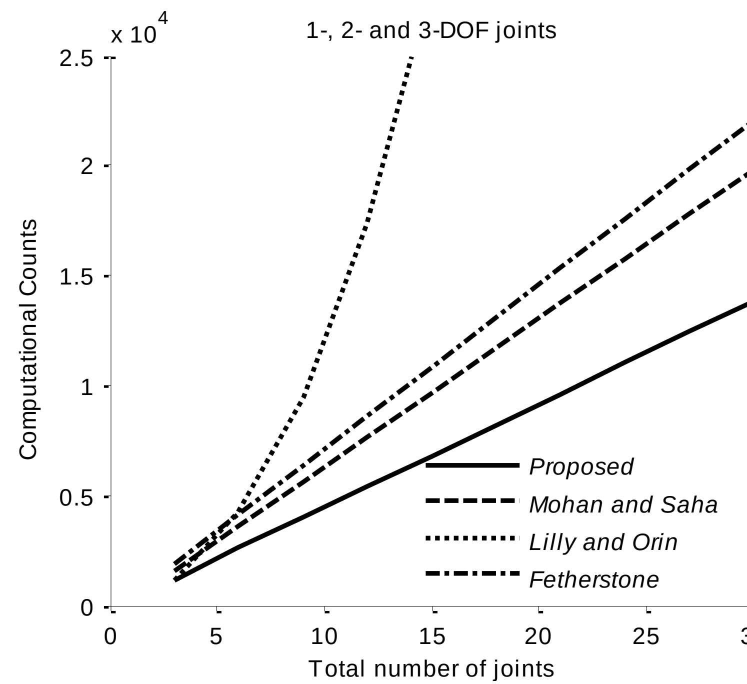
Why recursive?

- Efficient, i.e., less computations and CPU time

Compute jt. torque (Inverse)



Compute jt. accn. (Forward)

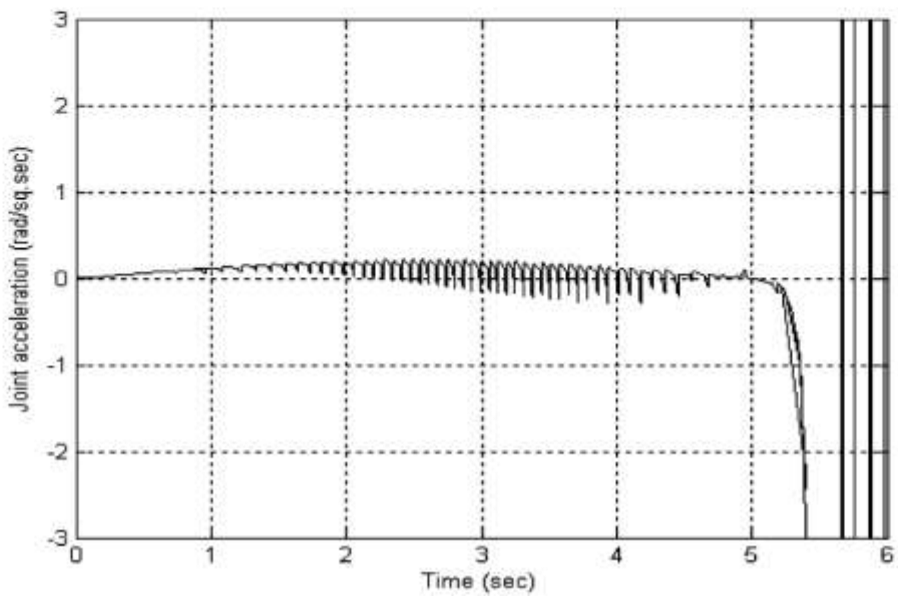


Ref. : Shah, S.,V. Saha, S.K., Dutt, J.K, Dynamics of Tree-type Robotic Systems, Springer, 2013

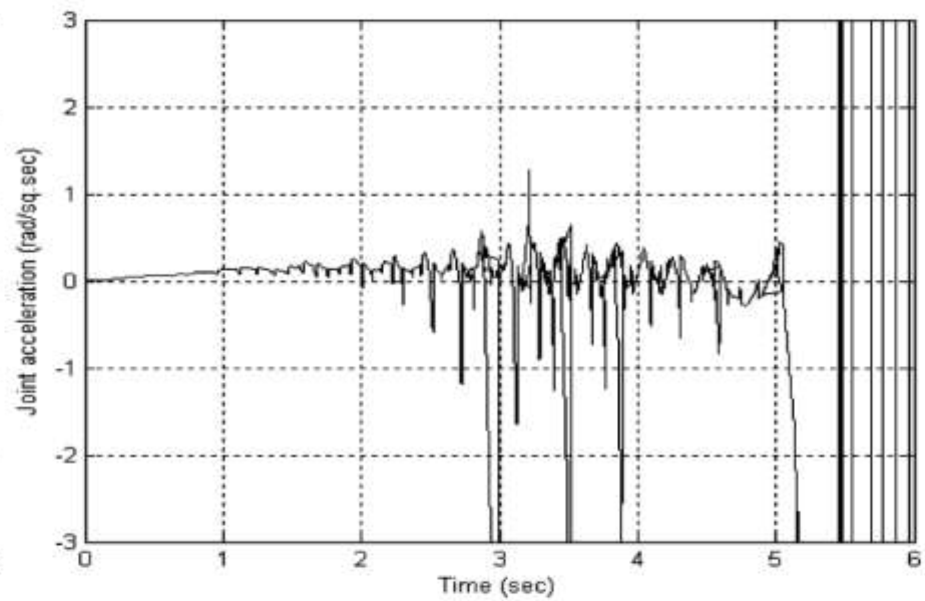
Why recursive? (contd.)

- Numerically stable → Simulation is realistic

Recursive (Forward)

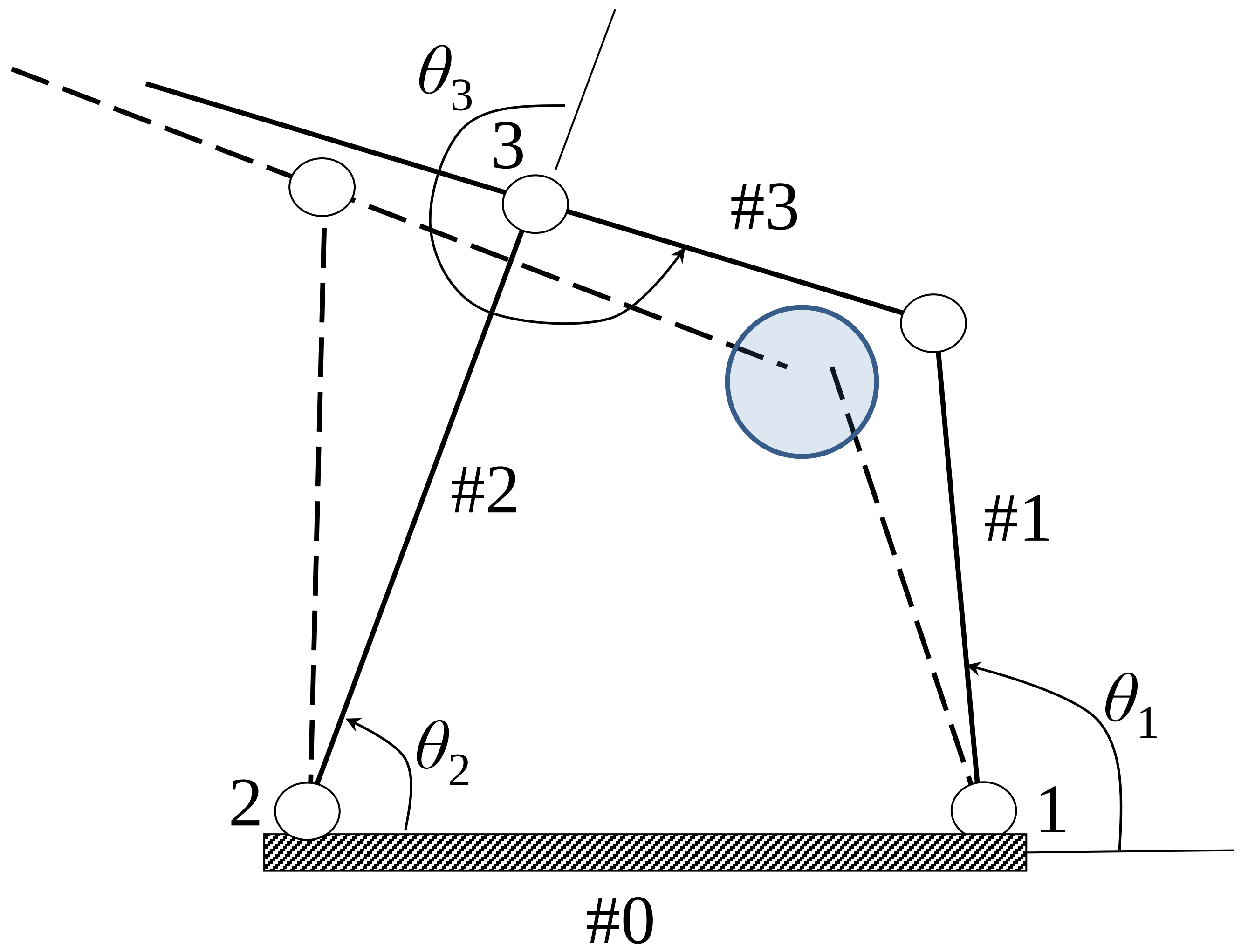


Non-recursive (Forward)



Ref. : Mohan, A., and **Saha, S.K.**, A recursive, numerically stable, and efficient simulation algorithm for serial robots with flexible links, Multibody System Dyn., V. 21, N. 1,pp. 1—35.

Unrealistic: Constraint Violation



Review of Dynamic Formulations

- Euler-Lagrange (EL)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

- Newton-Euler (NE)

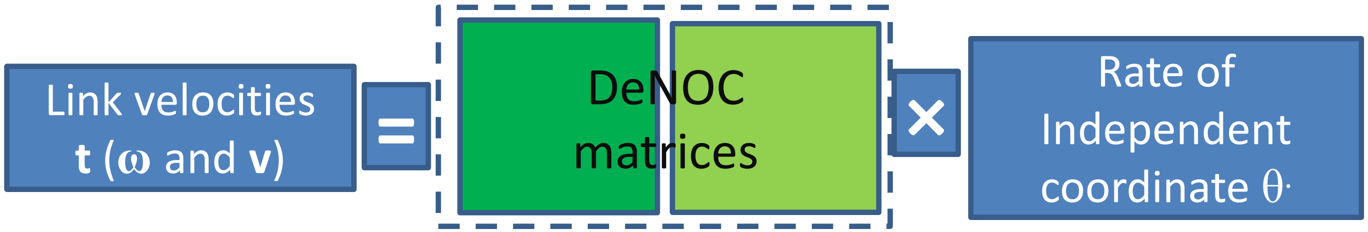
$$m_i \dot{\mathbf{v}}_i = \mathbf{f}_i$$
$$\mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i = \mathbf{n}_i$$

$$\mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{t}_i = \mathbf{w}_i$$

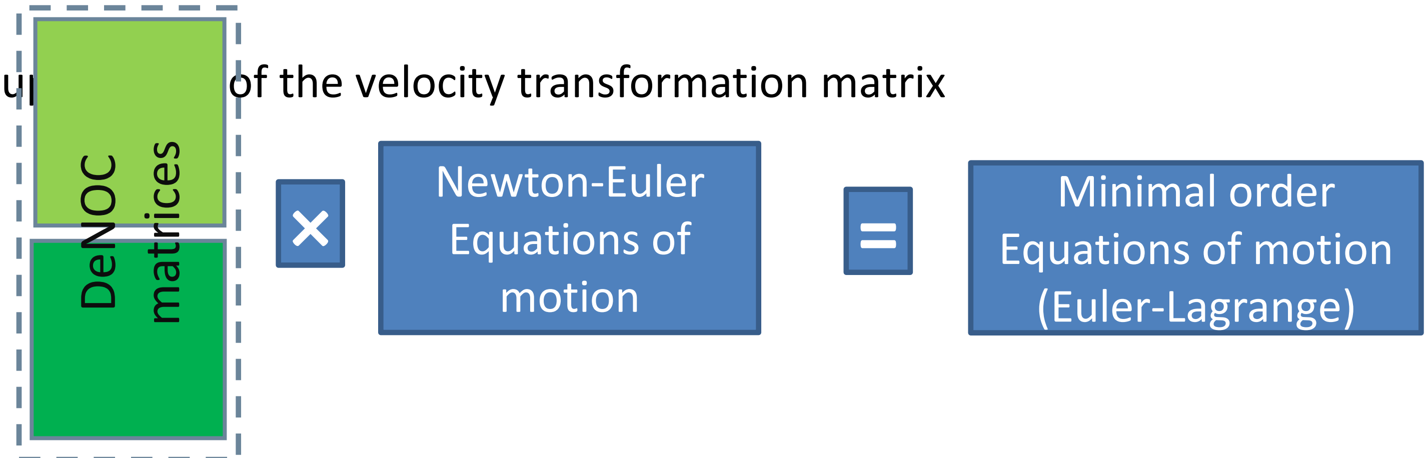
$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix}, \mathbf{W}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \times \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix}, \mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix}$$

Starting point for the Recursive Dynamics
(using the DeNOC matrices)

Newton-Euler to Euler-Lagrange



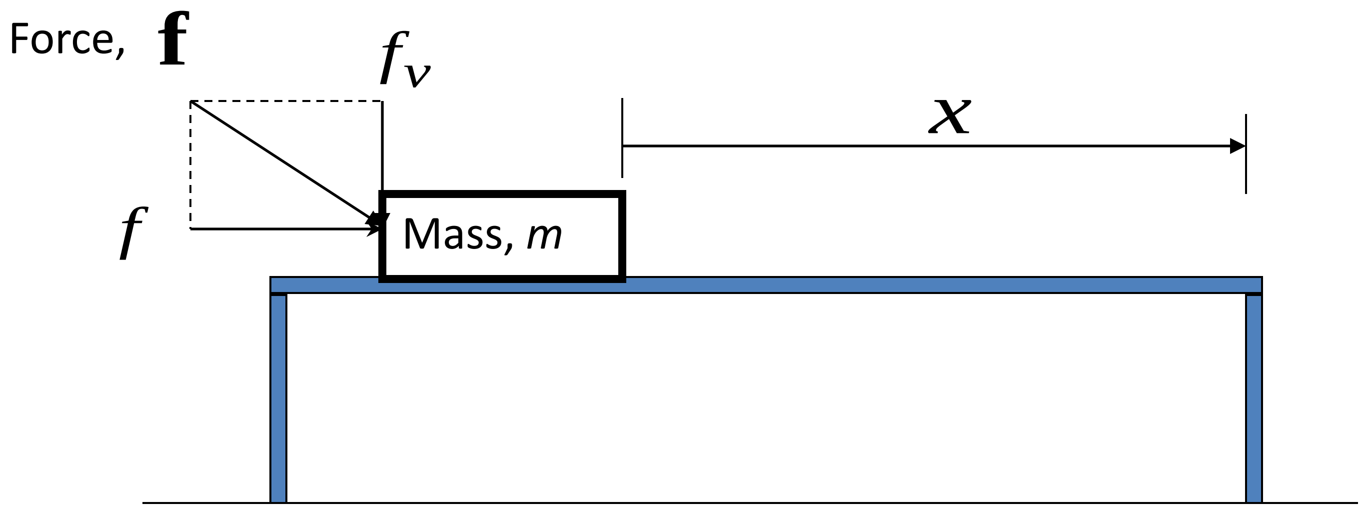
- Decou of the velocity transformation matrix



- Eliminate constraint forces and moments from the NE equations.

- Analytical expressions of vector and matrices, Decomposition of inertia Matrix, Recursive algorithms, Dynamics model simplifications, etc.

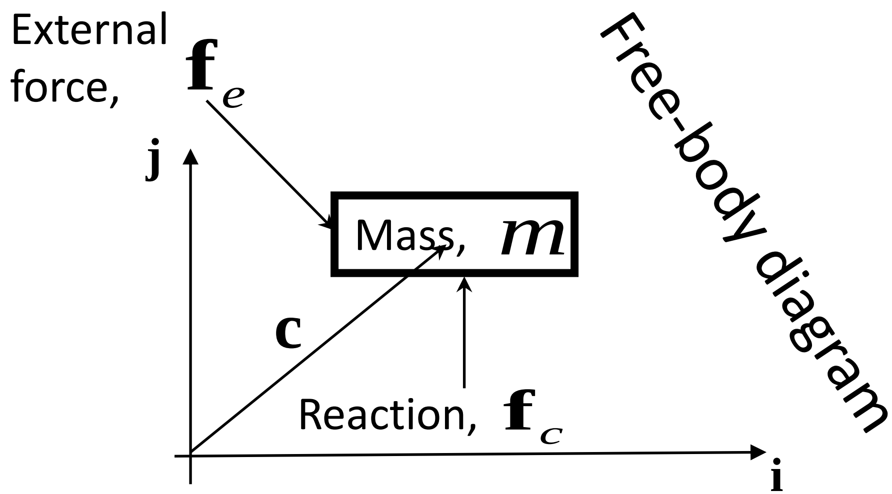
Example: A Moving Mass



f_v: Vertical component → Reaction

f : Horizontal component → Motion

Equation of Motion



Newton's 2nd law: $\mathbf{f}_e + \mathbf{f}_c = m\ddot{\mathbf{c}}$

Velocity constraint: $\dot{\mathbf{c}} = [\mathbf{i}]\dot{x}$

NOC: $[\mathbf{i}]$

Note that

$$[\mathbf{i}]^T (f_v + f_c)[\mathbf{j}] = 0$$

Euler-Lagrange:

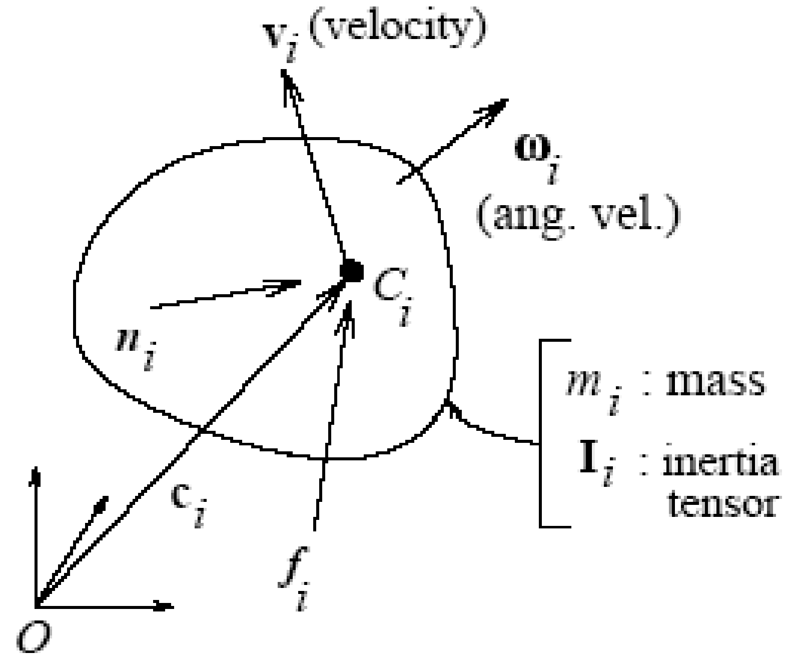
$$[\mathbf{i}]^T [f\mathbf{i} + (f_v + f_c)\mathbf{j}] = [\mathbf{i}]^T m\ddot{x}\mathbf{i} \Rightarrow f = m\ddot{x}$$

Uncoupled NE Equations

- Newton-Euler (NE) equations fo

$$\mathbf{I}_i\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i = \mathbf{n}_i$$

$$m_{\mathrm{i}}\dot{\mathbf{v}}_i = \mathbf{f}_i$$



- NE equations in compact form

$$\mathbf{M}_i\dot{\mathbf{t}}_i + \mathbf{W}_i\mathbf{M}_i\mathbf{t}_i = \mathbf{w}_i$$

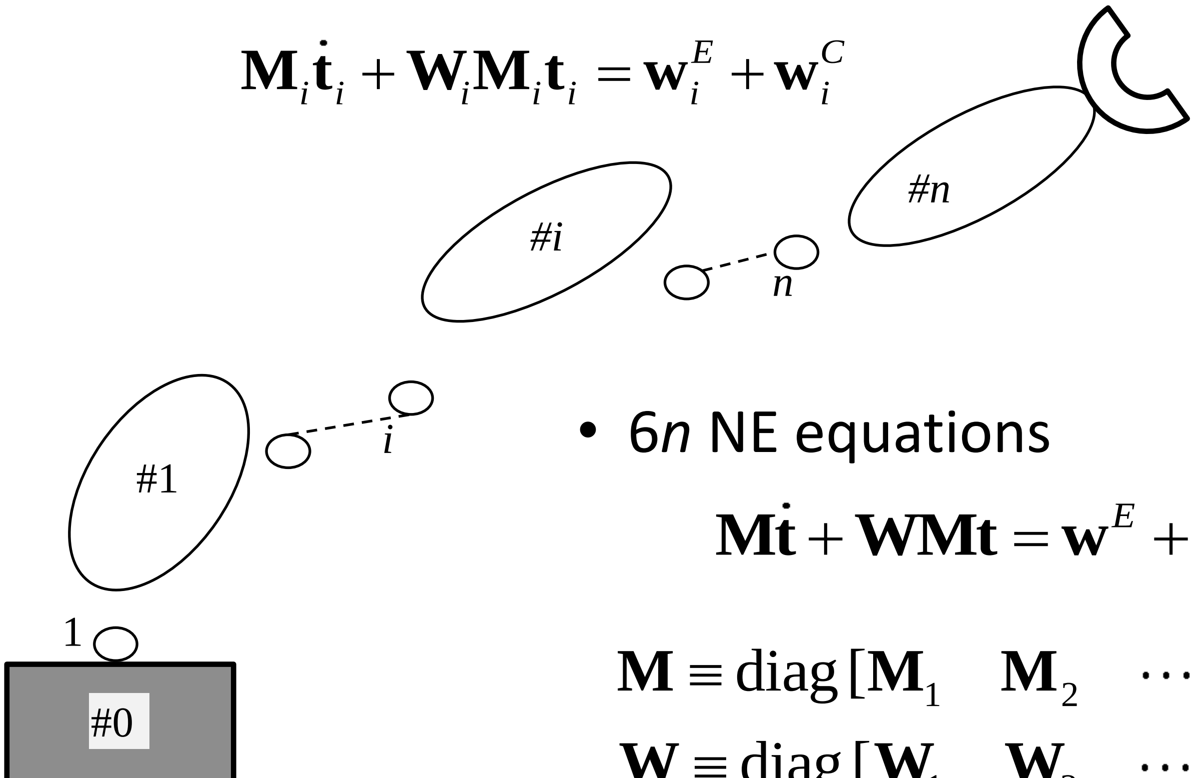
where

$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i\mathbf{1} \end{bmatrix}, \mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \dot{\mathbf{v}}_i \end{bmatrix}, \mathbf{W}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \times \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix}$$

Uncoupled NE Equations

- Separate the bodies → *n* bodies

$$\mathbf{M}_i\dot{\mathbf{t}}_i + \mathbf{W}_i\mathbf{M}_i\mathbf{t}_i = \mathbf{w}_i^E + \mathbf{w}_i^C$$



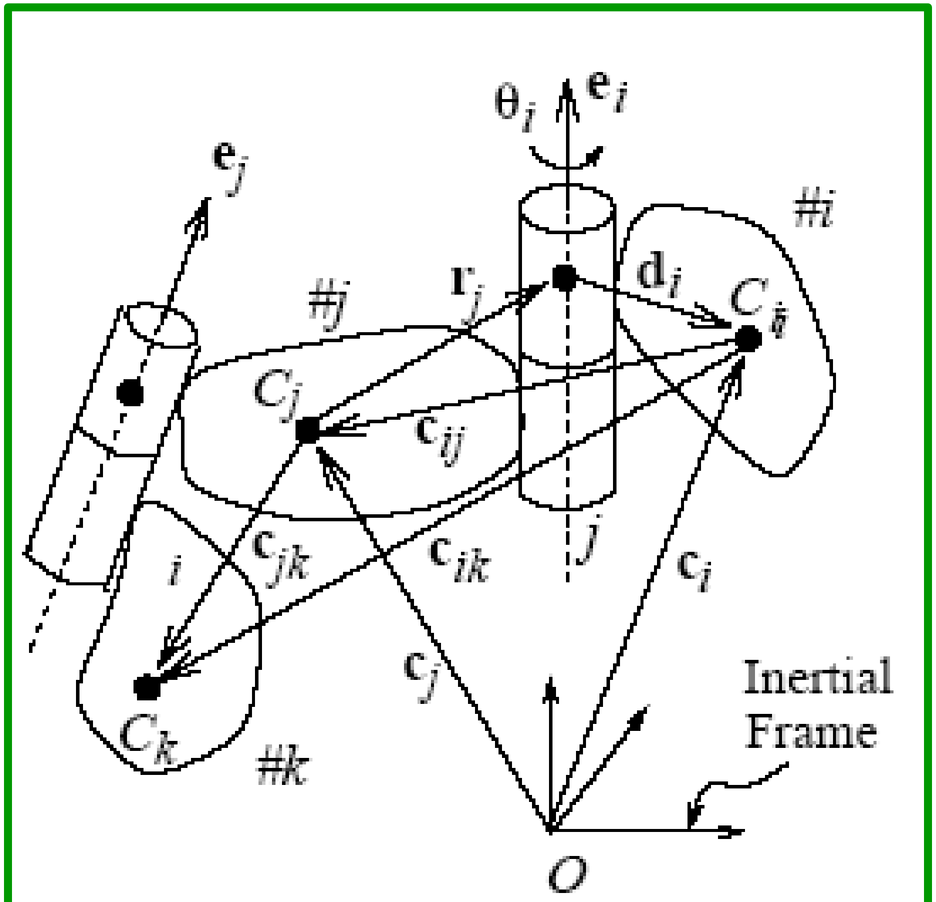
- 6*n* NE equations

$$\mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t} = \mathbf{w}^E + \mathbf{w}^C$$

$$\mathbf{M} \equiv \text{diag}[\mathbf{M}_1 \quad \mathbf{M}_2 \quad \cdots \quad \mathbf{M}_n]$$

$$\mathbf{W} \equiv \text{diag}[\mathbf{W}_1 \quad \mathbf{W}_2 \quad \cdots \quad \mathbf{W}_n]$$

Kinematic (Velocity) Constraints



$$\omega_i = \omega_j + \dot{\theta}_i \mathbf{e}_i$$

$$\mathbf{v}_i = \mathbf{v}_j + \omega_j \times \mathbf{r}_j + \omega_i \times \mathbf{d}_i$$

$$\mathbf{t}_i = \mathbf{B}_{ij}\mathbf{t}_j + \mathbf{p}_i\dot{\theta}_i$$

$$\mathbf{B}_{ij} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \underbrace{-(\mathbf{r}_j + \mathbf{d}_i)}_{\mathbf{c}_{ij}}\times\mathbf{1} & \mathbf{1} \end{bmatrix} \quad \mathbf{p}_i \equiv \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_i \times \mathbf{d}_i \end{bmatrix}$$

$$\mathbf{B}_{ij}\mathbf{B}_{jk} = \mathbf{B}_{ik}$$

$$\mathbf{B}_{ii} = \mathbf{1}, \quad \text{and} \quad \mathbf{B}_{ij}^{-1} = \mathbf{B}_{ji}$$

$\mathbf{B}_{ij}$: the $6n \times 6n$ <u>twist-propagation</u> matrix

$\mathbf{p}_i$: the $6n$-dimensional joint-rate propagation vector or <u>twist generator</u>

DeNOC Matrices

$$\mathbf{t} \equiv [\mathbf{t}_1^T, \cdots, \mathbf{t}_n^T]^T \quad \dot{\boldsymbol{\theta}} \equiv [\dot{\theta}_1, \cdots, \dot{\theta}_n]^T$$

$$\mathbf{t} = \mathbf{N}\dot{\boldsymbol{\theta}}, \qquad \text{where} \quad \mathbf{N} \equiv \mathbf{N}_l \mathbf{N}_d$$

$$\mathbf{N}_l \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{B}_{21} & \mathbf{1} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n1} & \mathbf{B}_{n2} & \cdots & \mathbf{1} \end{bmatrix} \quad \text{and} \quad \mathbf{N}_d \equiv \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{p}_n \end{bmatrix}$$

- **$\mathbf{N} \equiv \mathbf{N}_l \mathbf{N}_d$:** the $6n \times n$ Decoupled Natural Orthogonal Complement

Coupled Equations of Motion

- Pre-multiplication by $\mathbf{N}^T$

$$\mathbf{N}^T(\mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t}) = \mathbf{N}^T(\mathbf{w}^E + \mathbf{w}^C)$$

- Equations in compact form

$$\because \mathbf{N}^T\mathbf{w}^C = \mathbf{0} \qquad \mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \boldsymbol{\tau}$$

n coupled EL equations
- *no partial differentiation*

I : $n \times n$ Generalized inertia matrix (GIM)

C : $n \times n$ Matrix of convective inertia (MCI) terms

$\boldsymbol{\tau}$: n-dimensional vector of generalized forces due to driving torques/forces, and those resulting from the gravity, environment and dissipation.

Generalized Inertia Matrix (GIM)

- Generalized inertia matrix (GIM)

$$\mathbf{I} = \mathbf{N}_d^T \tilde{\mathbf{M}} \mathbf{N}_d \quad \text{where} \quad \tilde{\mathbf{M}} \equiv \mathbf{N}_l^T \mathbf{M} \mathbf{N}_l$$

- Each element of the GIM

$$i_{ij} = \mathbf{p}_i^T \tilde{\mathbf{M}}_i \mathbf{B}_{ij} \mathbf{p}_j$$

- Mass matrix of composite body

$$\tilde{\mathbf{M}}_i = \mathbf{M}_i + \mathbf{B}_{i+1,i}^T \tilde{\mathbf{M}}_{i+1} \mathbf{B}_{i+1,i}$$

Vector of Convective Inertia (VCI)

- Vector of Convective Inertia

$$\mathbf{h} \equiv \mathbf{C}\dot{\boldsymbol{\theta}} = \mathbf{N}_d^T \tilde{\mathbf{w}}'$$

where $\tilde{\mathbf{w}}' = \mathbf{N}_d^T(\mathbf{M}\mathbf{t}' + \mathbf{W}\mathbf{M}\mathbf{t})$ and $\mathbf{t}' = (\dot{\mathbf{N}}_l + \mathbf{N}_l\mathbf{W})\dot{\boldsymbol{\theta}}$

- Each element of **h**

$$\mathbf{h}_i = \mathbf{p}_i^T \tilde{\mathbf{w}}_i'$$

where $\tilde{\mathbf{w}}_i' = \mathbf{w}_i' + \mathbf{B}_{i+1,i}^T \tilde{\mathbf{w}}_{i+1}'$, and $\tilde{\mathbf{w}}_n' = \mathbf{w}_n'$

and $\mathbf{w}_i' \equiv \mathbf{M}_i\mathbf{t}_i' + \mathbf{W}_i\mathbf{M}_i\mathbf{t}_i$

Generalized Force (Joint Torque)

- Generalized Force

$$\boldsymbol{\tau} = \mathbf{N}_d^T \tilde{\mathbf{w}}^E \quad \text{where} \quad \tilde{\mathbf{w}}^E = \mathbf{N}_l^T \mathbf{w}^E$$

- Each element is obtained recursively

$$\tau_i = \mathbf{p}_i^T \tilde{\mathbf{w}}_i^E,$$

$$\text{where } \tilde{\mathbf{w}}_i^E = \mathbf{w}_i^E + \mathbf{B}_{i+1,i}^T \tilde{\mathbf{w}}_{i+1}^E, \text{ and } \tilde{\mathbf{w}}_n^E = \mathbf{w}_n^E$$

;

Example: One-link arm

$$[\mathbf{e}]_1 \equiv [0 \quad 0 \quad 1]^T ; [\mathbf{d}]_1 \equiv [\frac{1}{2} ac\theta \quad \frac{1}{2} as\theta \quad 0]^T$$

$$[\mathbf{I}]_2 = \frac{ma^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{I}]_1 = \mathbf{Q}[\mathbf{I}]_2\mathbf{Q}^T = \frac{ma^2}{12}\begin{bmatrix} s^2\theta & -s\theta c\theta & 0 \\ -s\theta c\theta & c^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{p} \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{e} \times \mathbf{d} \end{bmatrix}$ and $\tilde{\mathbf{M}} \equiv \mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & m\mathbf{1} \end{bmatrix}$

$$I(\equiv i_{11}) = \mathbf{p}^T \tilde{\mathbf{M}} \mathbf{p}$$

$$= \mathbf{e}^T \mathbf{I} \mathbf{e} + m(\mathbf{e} \times \mathbf{d})^T (\mathbf{e} \times \mathbf{d}) = \frac{1}{3} ma^2$$

Moment of inertia about *O*

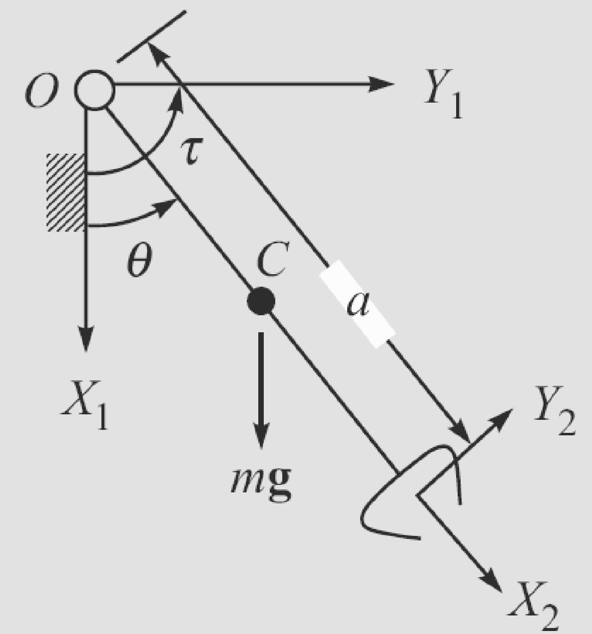


Fig. 9.3 One-link arm

Ref: "**Introduction to Robotics**" by Saha

$$h = \mathbf{p}^T(\mathbf{MW} + \mathbf{WM})\mathbf{p}$$

$$= \dot{\theta}\mathbf{e}^T[\mathbf{I}(\mathbf{e}\times\mathbf{e}) + (\mathbf{e}\times\mathbf{Ie})] = 0$$

$$\tau_1 = \mathbf{N}_l^T\mathbf{w}^E = [\mathbf{e}^T \quad (\mathbf{e}\times\mathbf{d})^T]\begin{bmatrix}\mathbf{n}\\ \mathbf{f}\end{bmatrix} \Rightarrow \tau_1 = \tau - \frac{1}{2}mgas\theta$$

where $[\mathbf{n}]_1 \equiv [0 \quad 0 \quad \tau]^T; [\mathbf{f}]_1 = [mg \quad 0 \quad 0]^T$

Equation of motion:

$$\frac{1}{3}ma^2\ddot{\theta} = \tau - \frac{1}{2}mgas\theta$$

Recursive Inverse Dynamics

$$\boldsymbol{\gamma}_n = \mathbf{M}_n \boldsymbol{\beta}_n + \mathbf{W}_n \mathbf{M}_n \boldsymbol{\alpha}_n$$

$$\boldsymbol{\gamma}_{n-1} = \mathbf{M}_{n-1} \boldsymbol{\beta}_{n-1} + \mathbf{W}_{n-1} \mathbf{M}_{n-1} \boldsymbol{\alpha}_{n-1} + \mathbf{B}_{n,n-1}^T \boldsymbol{\gamma}_n$$

$$\vdots$$

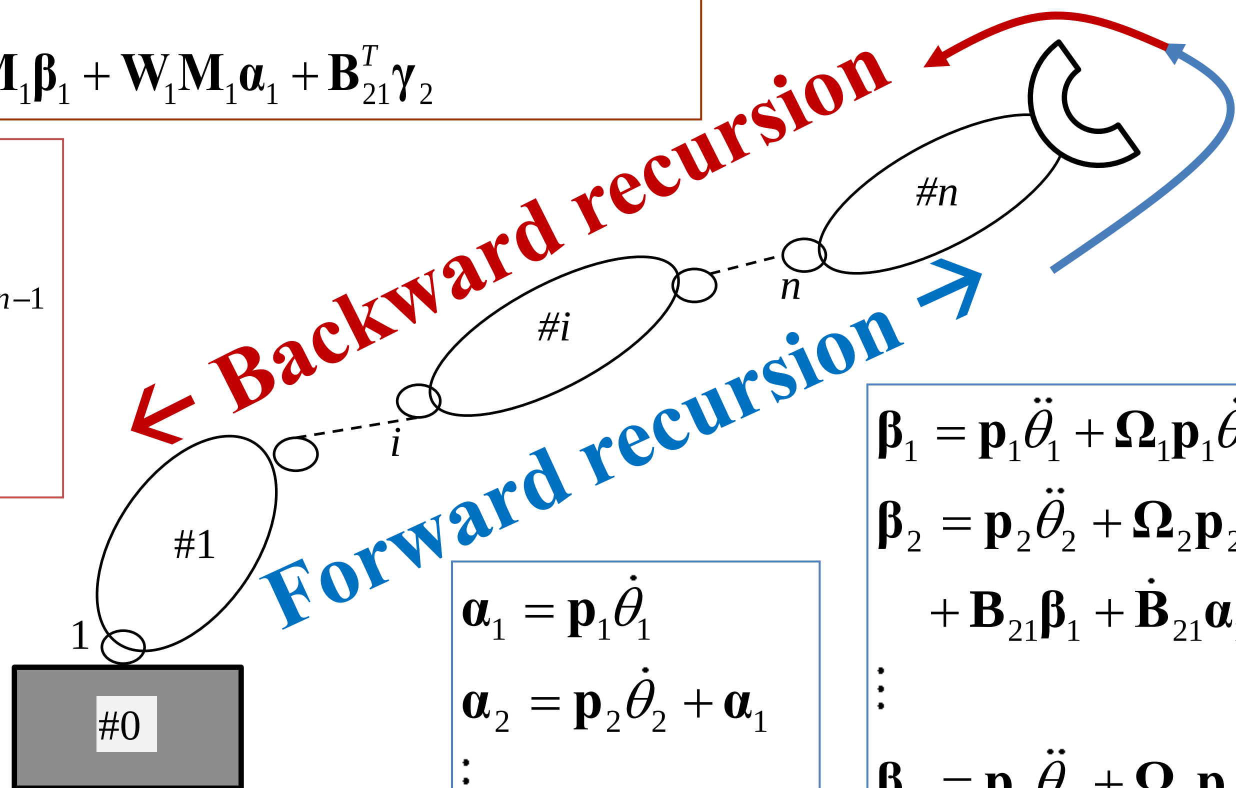
$$\boldsymbol{\gamma}_1 = \mathbf{M}_1 \boldsymbol{\beta}_1 + \mathbf{W}_1 \mathbf{M}_1 \boldsymbol{\alpha}_1 + \mathbf{B}_{21}^T \boldsymbol{\gamma}_2$$

$$\tau_n = \mathbf{p}_n^T \boldsymbol{\gamma}_n$$

$$\tau_{n-1} = \mathbf{p}_{n-1}^T \boldsymbol{\gamma}_{n-1}$$

$$\vdots$$

$$\tau_1 = \mathbf{p}_1^T \boldsymbol{\gamma}_1$$



$$\boldsymbol{\alpha}_1 = \mathbf{p}_1 \dot{\theta}_1$$

$$\boldsymbol{\alpha}_2 = \mathbf{p}_2 \dot{\theta}_2 + \boldsymbol{\alpha}_1$$

$$\vdots$$

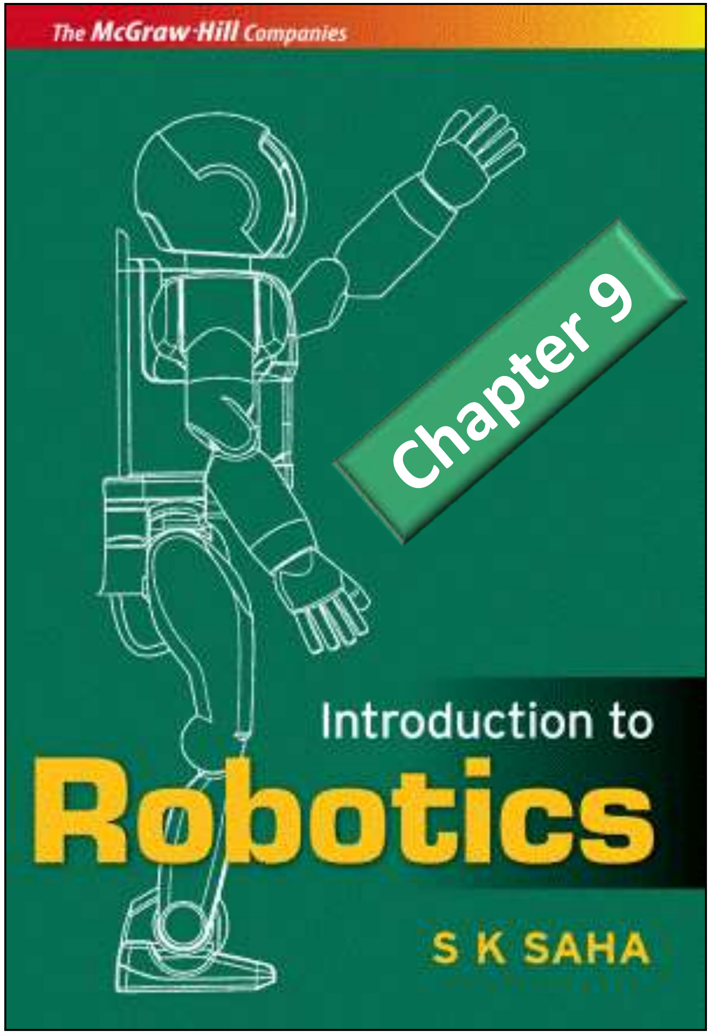
$$\boldsymbol{\alpha}_n = \mathbf{p}_n \dot{\theta}_n + \boldsymbol{\alpha}_{n-1}$$

$$\boldsymbol{\beta}_1 = \mathbf{p}_1 \ddot{\theta}_1 + \boldsymbol{\Omega}_1 \mathbf{p}_1 \dot{\theta}_1$$

$$\boldsymbol{\beta}_2 = \mathbf{p}_2 \ddot{\theta}_2 + \boldsymbol{\Omega}_2 \mathbf{p}_2 \dot{\theta}_2 + \mathbf{B}_{21} \boldsymbol{\beta}_1 + \dot{\mathbf{B}}_{21} \boldsymbol{\alpha}_1$$

$$\vdots$$

$$\boldsymbol{\beta}_n = \mathbf{p}_n \ddot{\theta}_n + \boldsymbol{\Omega}_n \mathbf{p}_n \dot{\theta}_n + \mathbf{B}_{n,n-1} \boldsymbol{\beta}_{n-1} + \dot{\mathbf{B}}_{n,n-1} \boldsymbol{\alpha}_{n-1}$$



Lecture 2

Recursive Robot Dynamics

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Summary of Previous Lecture

- Why recursive?
- Define DeNOC (Decoupled Natural Orthogonal Complement) matrices from velocity constraints
- Derived NE → EL equations (Constrained minimum set)
- Analytical expressions for the GIM, and other matrices

One-link Arm

Equation of motion
(Dynamic Model)

$$\frac{1}{3}ma^2\ddot{\theta} = \tau - \frac{1}{2}mgas\theta$$

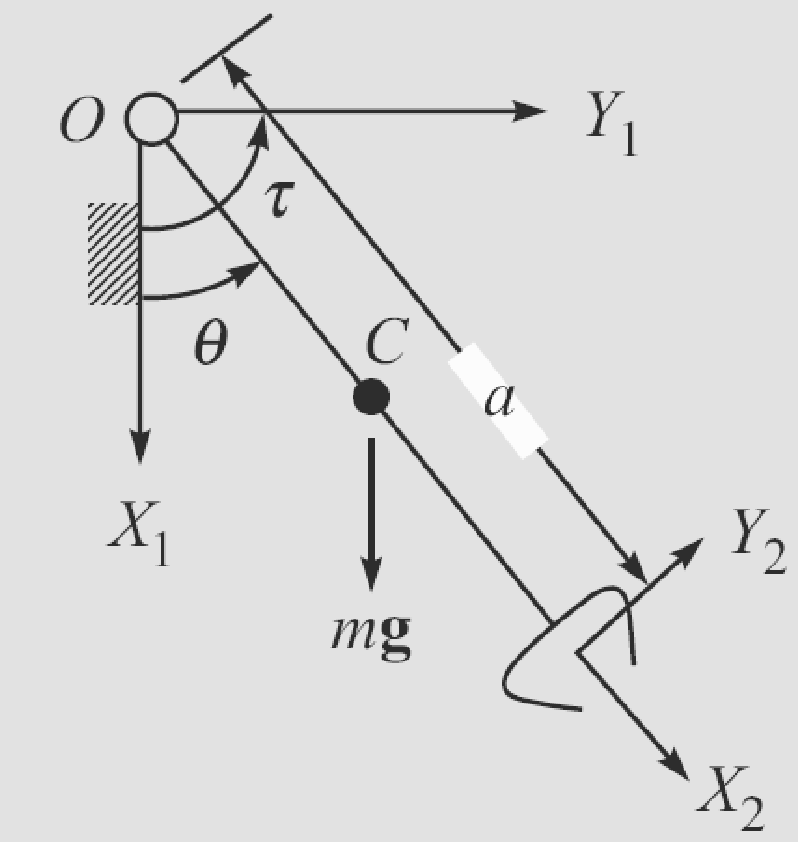
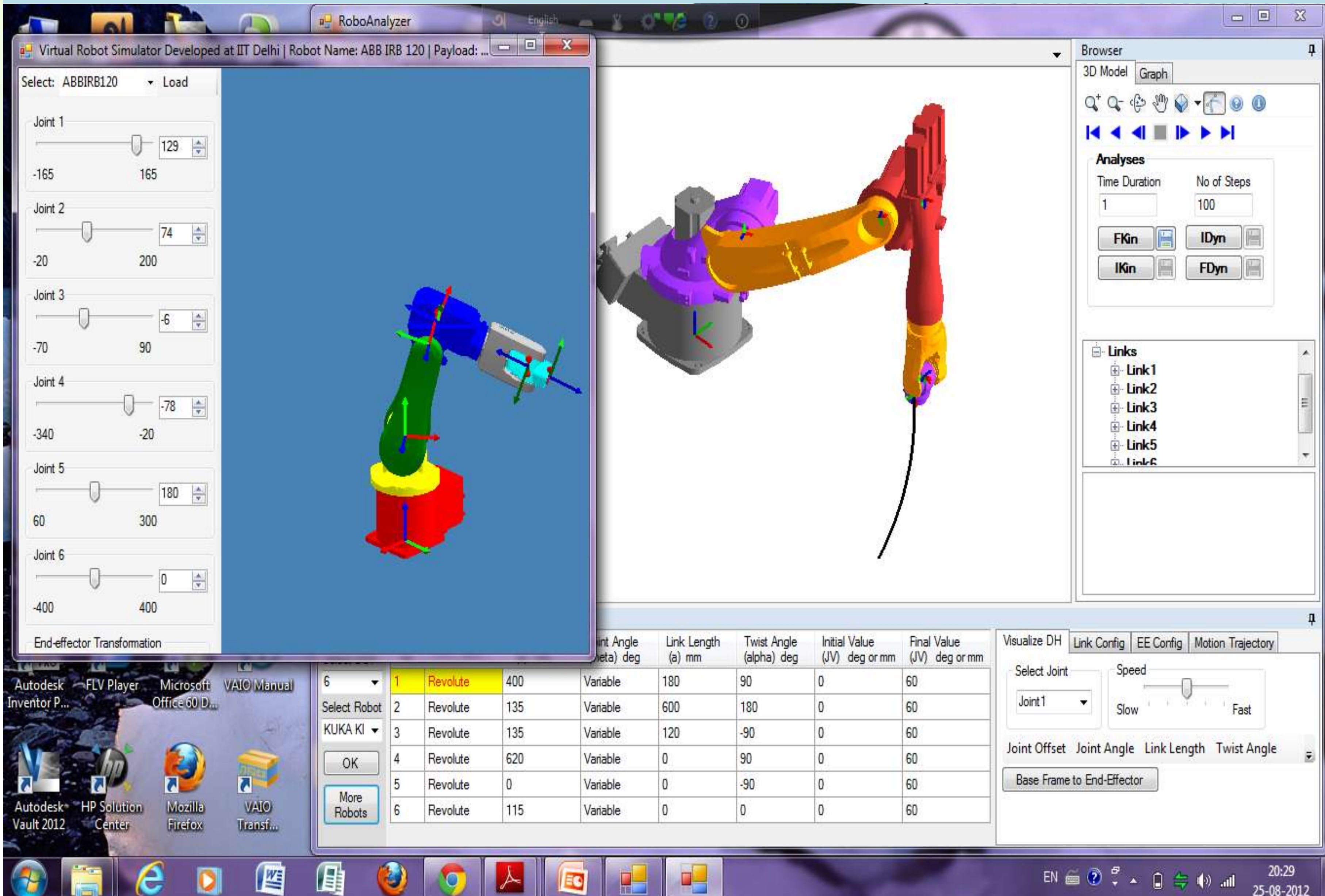


Fig. 9.3 One-link arm

RoboAnalyzer (Free: www.roboanalyzer.com)



Verify the results using MATLAB's symbolic tool (MuPAD)

$$I(\equiv i_{11}) = \mathbf{p}^T \tilde{\mathbf{M}} \mathbf{p} = \mathbf{e}^T \mathbf{I} \mathbf{e} + m(\mathbf{e} \times \mathbf{d})^T (\mathbf{e} \times \mathbf{d}) = \frac{1}{3} m a^2$$

$$h = \mathbf{p}^T (\mathbf{MW} + \mathbf{WM}) \mathbf{p} = \dot{\theta} \mathbf{e}^T [\mathbf{I}(\mathbf{e} \times \mathbf{e}) + (\mathbf{e} \times \mathbf{I} \mathbf{e})] = 0$$

$$\tau_1 = \mathbf{N}_l^T \mathbf{w}^E = [\mathbf{e}^T \quad (\mathbf{e} \times \mathbf{d})^T] \begin{bmatrix} \mathbf{n} \\ \mathbf{f} \end{bmatrix} \Rightarrow \tau_1 = \tau - \frac{1}{2} m g a s\theta$$

where $[\mathbf{n}]_1 \equiv [0 \quad 0 \quad \tau]^T ; [\mathbf{f}]_1 = [mg \quad 0 \quad 0]^T$

Equation of motion: $$\boxed{\frac{1}{3} m a^2 \ddot{\theta} = \tau - \frac{1}{2} m g a s\theta}$$

;

Example: Two-link Manipulator

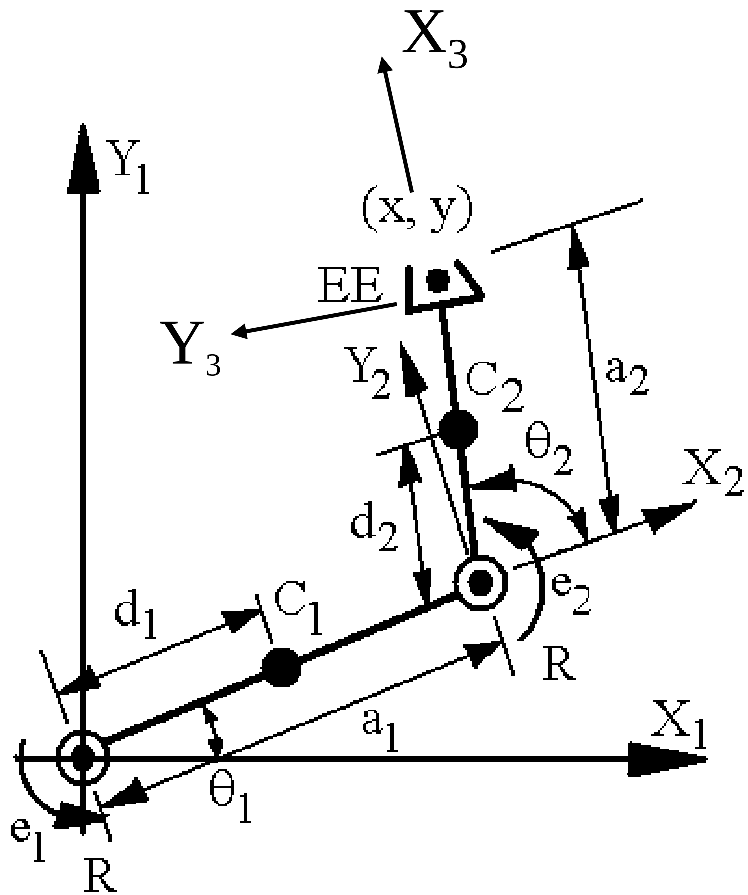
$$\mathbf{I} \equiv \begin{bmatrix} i_{11} & i_{12}(=i_{21}) \\ i_{21} & i_{22} \end{bmatrix}; \ \mathbf{h} \equiv \mathbf{C}\dot{\boldsymbol{\theta}} \equiv \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}; \boldsymbol{\tau} \equiv \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$i_{22} \equiv \mathbf{p}_2^T \tilde{\mathbf{M}}_2 \mathbf{B}_{22} \mathbf{p}_2 : \text{Scalar}$$

$$\mathbf{p}_2 \equiv \begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_2 \times \mathbf{d}_2 \end{bmatrix} : \text{6-dim. vector}$$

$$\tilde{\mathbf{M}}_2 \equiv \mathbf{M}_2 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & m_2 \mathbf{1} \end{bmatrix} : \ 6 \times 6 \text{ sym. matrix}$$

$$\mathbf{B}_{22} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{bmatrix} : 6 \times 6 \text{ identity matrix}$$



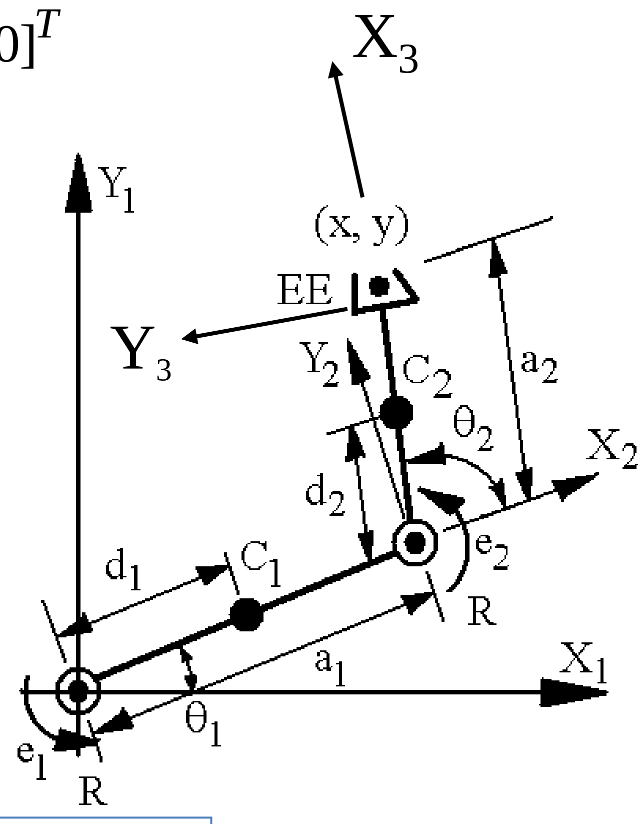
$$i_{22} = [\mathbf{e}_2]_2^T [\mathbf{I}_2]_2 [\mathbf{e}_2]_2 + m_2 [\mathbf{d}_2]_2^T [\mathbf{d}_2]_2$$

$$i_{22} = [\mathbf{e}_2]_2^T [\mathbf{I}_2]_2 [\mathbf{e}_2]_2 + m_2 [\mathbf{d}_2]_2^T [\mathbf{d}_2]_2$$

$$[\mathbf{e}_2]_2 \equiv [0 \quad 0 \quad 1]^T; \; [\mathbf{d}_2]_2 \equiv [\frac{1}{2} a_2 c\theta_2 \quad \frac{1}{2} a_2 s\theta_2 \quad 0]^T$$

$$\mathbf{Q}_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{I}_2]_2 = \mathbf{Q}_2 [\mathbf{I}_2]_3 \mathbf{Q}_2^T = \frac{ma^2}{12} \begin{bmatrix} s^2\theta_2 & -s\theta_2 c\theta_2 & 0 \\ -s\theta_2 c\theta_2 & c^2\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$i_{22} = \frac{1}{12} m_2 a_2^2 + \frac{1}{4} m_2 a_2^2 = \frac{1}{3} m_2 a_2^2$$

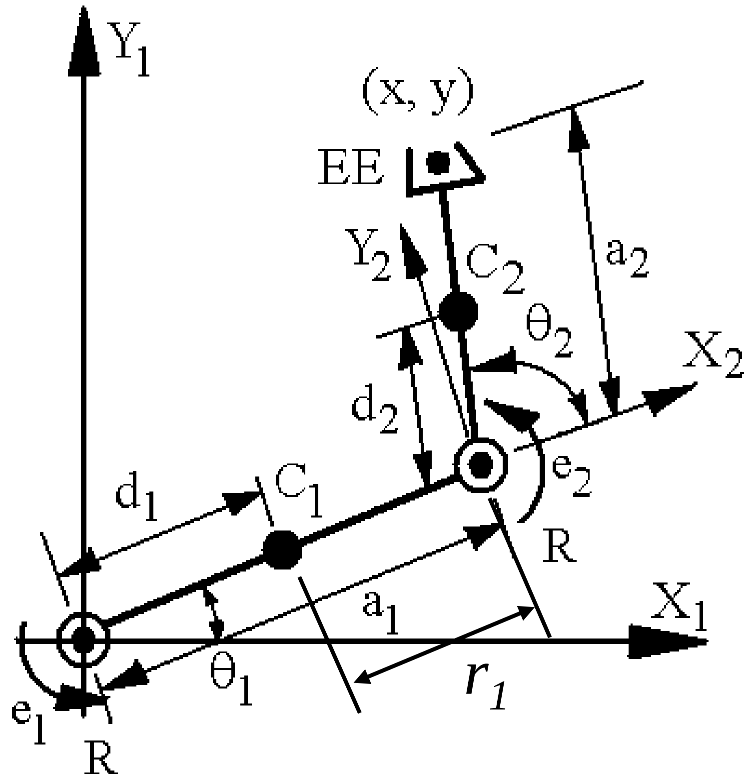
;

$$i_{21}(=i_{12}) \equiv \mathbf{p}_2^T \tilde{\mathbf{M}}_2 \mathbf{B}_{21} \mathbf{p}_1 \text{ : Scalar}$$

$$\mathbf{B}_{21} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -(\mathbf{r}_1 + \mathbf{d}_2) \times \mathbf{1} & \mathbf{1} \end{bmatrix} : 6 \times 6 \text{ matrix}$$

$$\mathbf{p}_1 \equiv \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_1 \times \mathbf{d}_1 \end{bmatrix} : \text{6-dim. vector}$$

$$\mathbf{Q}_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$i_{21} = [\mathbf{e}_2]_1^T [\mathbf{I}_2]_1 [\mathbf{e}_1]_1 + m_2 [\mathbf{d}_2]_1^T ([\mathbf{d}_1 + \mathbf{r}_1]_1 + [\mathbf{d}_2]_1)$$

$$[\mathbf{e}_2]_1 \equiv [0 \quad 0 \quad 1]^T; \ [\mathbf{d}_2]_1 = \mathbf{Q}_1 [\mathbf{d}_2]_2 = [\frac{1}{2} a_2 c\theta_{12} \quad \frac{1}{2} a_2 s\theta_{12} \quad 0]^T$$

$$[\mathbf{e}_1]_1 \equiv [0 \quad 0 \quad 1]^T; \ [\mathbf{d}_1]_1 = [\mathbf{r}_1]_1 = [\frac{1}{2} a_1 c\theta_1 \quad \frac{1}{2} a_2 s\theta_1 \quad 0]^T$$

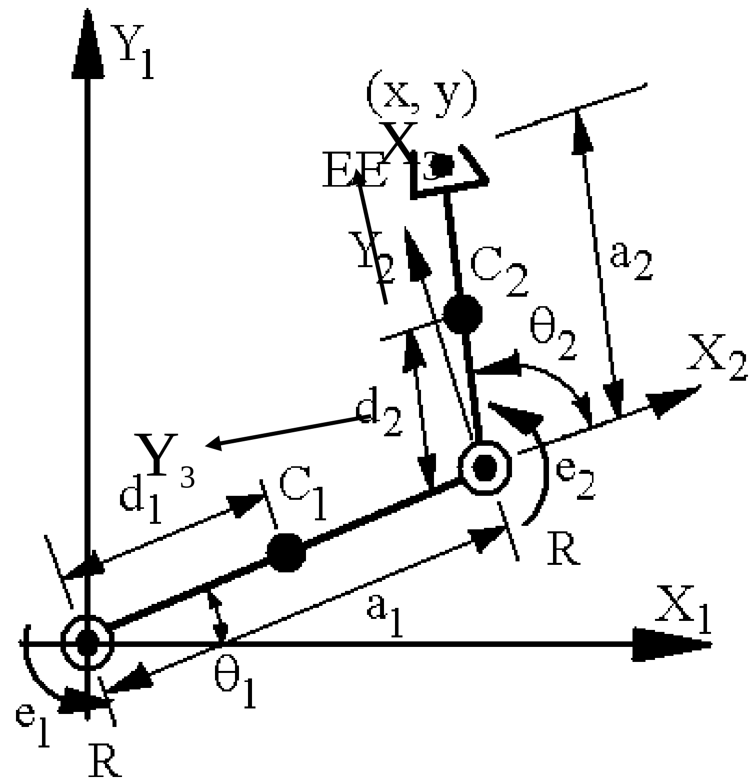
$$i_{21} = \frac{1}{12} m_2 a_2^2 + \frac{1}{2} m_2 a_1 a_2 c\theta_2 + \frac{1}{4} m_2 a_2^2 = \frac{1}{3} m_2 a_2^2 + \frac{1}{2} m_2 a_1 a_2 c\theta_2$$

$$i_{11} \equiv \mathbf{p}_1^T \tilde{\mathbf{M}}_1 \mathbf{B}_{11} \mathbf{p}_1 \text{ : Scalar}$$

$$\mathbf{B}_{11} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{bmatrix} : 6\times 6 \text{ identity matrix}$$

$$\mathbf{M}_1 \equiv \begin{bmatrix} \mathbf{I}_1 & \mathbf{O} \\ \mathbf{O} & m_1\mathbf{1} \end{bmatrix} : \ 6\times 6 \text{ sym. matrix}$$

$$\tilde{\mathbf{M}}_1 = \mathbf{M}_1 + \mathbf{B}_{21}^T \tilde{\mathbf{M}}_2 \mathbf{B}_{21} = \begin{bmatrix} \tilde{\mathbf{I}}_1 & \boldsymbol{\delta}_1 \times \mathbf{1} \\ -\boldsymbol{\delta}_1 \times \mathbf{1} & \tilde{m}_1 \mathbf{1} \end{bmatrix}$$



$$\tilde{\mathbf{I}}_1 = \mathbf{I}_1 + \mathbf{I}_2 + m_2 (\underbrace{\mathbf{r}_1 + \mathbf{d}_2}_{\mathbf{c}_{12}(=-\mathbf{c}_{21})}) \times \left(\tilde{\boldsymbol{\delta}}_1 \times \mathbf{1} \right)$$

$$\tilde{\boldsymbol{\delta}}_1 = m_2 \mathbf{c}_{21}$$

$$\tilde{m}_1 = m_1 + m_2$$

$$\begin{aligned} i_{11} &= [\mathbf{e}_1]_1^T [\tilde{\mathbf{I}}_1]_1 [\mathbf{e}\ \ ]_1 + \tilde{m}_1 [\mathbf{d}_1]_1^T [\mathbf{d}_1]_1 \\ &= \frac{1}{3}(m_1 a_1^2 + m_2 a_2^2) + m_2 a_1^2 + m_2 a_1 a_2 c\theta_2 \end{aligned}$$

Vector of Convective Inertia

$$h_2 = \mathbf{p}_2^T \tilde{\mathbf{w}}'_2 = \frac{1}{2} m_2 a_1 a_2 s\theta_2 \dot{\theta}_1^2$$

$$h_1 = \mathbf{p}_1^T \tilde{\mathbf{w}}'_1 = -m_2 a_1 a_2 s\theta_2 \dot{\theta}_2 (\frac{1}{2}\dot{\theta}_2 + \dot{\theta}_1)$$

Inverse Dynamics Results

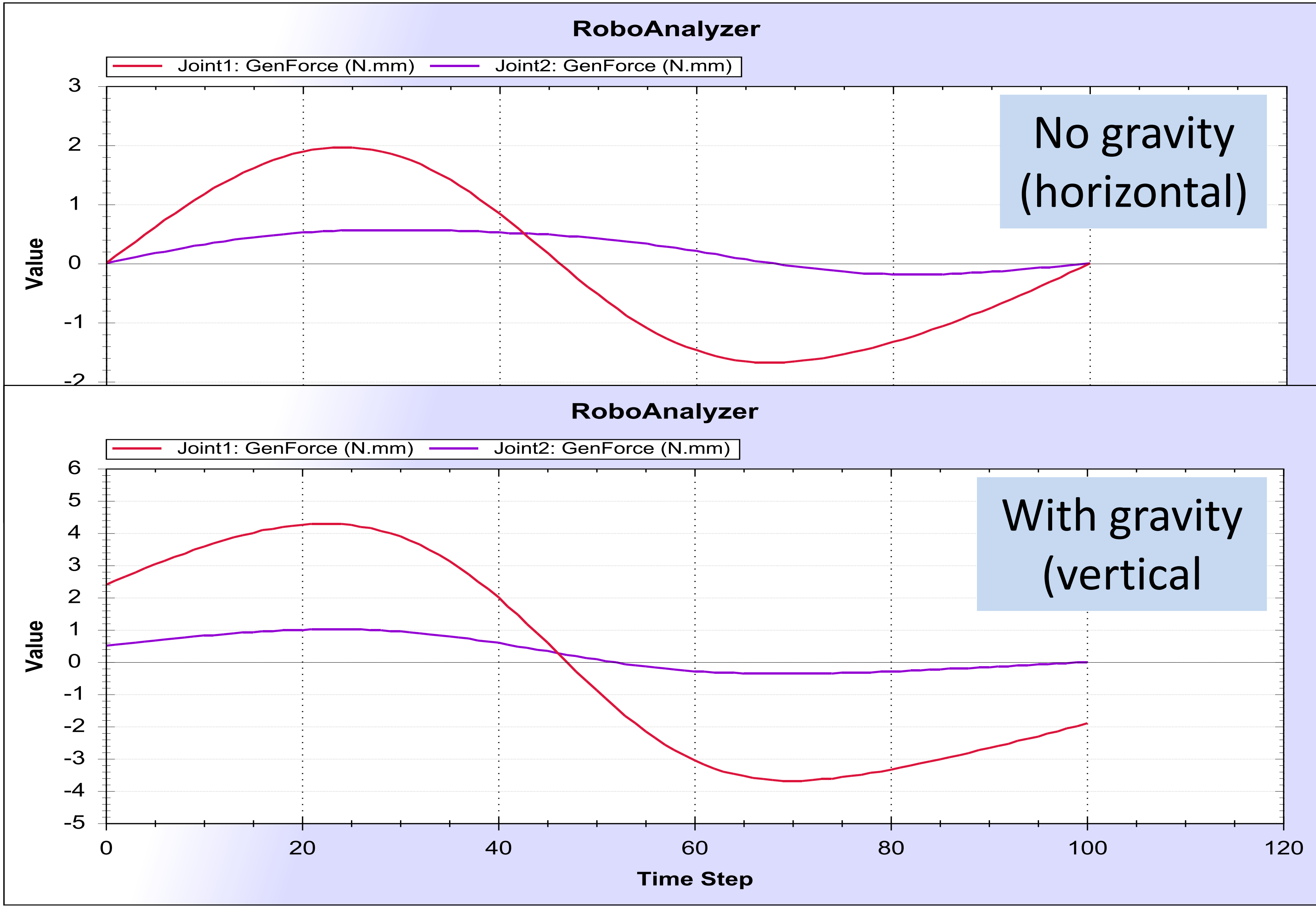
Using RoboAnalyzer

DH and Inertia parameters

Link	Joint	a_i (m)	b_i (m)	α_i (rad)	θ_i (rad)
1	r	0.3	0	0	JV [0]
2	r	0.25	0	0	JV [0]

Link	m_i	$r_{i,x}$	$r_{i,y}$	$r_{i,z}$	$I_{i,xx}$	$I_{i,xy}$	$I_{i,xz}$	$I_{i,yy}$	$I_{i,yz}$	$I_{i,zz}$
	(kg)	(m)			(kg-m^2)					
1	0.5	0.15	0	0	0	0	0	0.00375	0	0.00375
2	0.4	0.125	0	0	0	0	0	0.00208	0	0.00208

Joint Torques



Forward Dynamics & Simulation

Equation of motion for one-link arm

$$\frac{1}{3}ma^2\ddot{\theta} = \tau - \frac{1}{2}mgas\theta$$

Forward Dynamics:

$$\ddot{\theta} = \frac{3}{ma^2}(\tau - \frac{1}{2}mgas\theta)$$

+

Integration (numerical):

$$y_1 = \theta; y_2 = \dot{\theta}$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = \ddot{\theta} = \frac{3}{ma^2}(\tau - \frac{1}{2}mga\sin y_1)$$

1st order form:

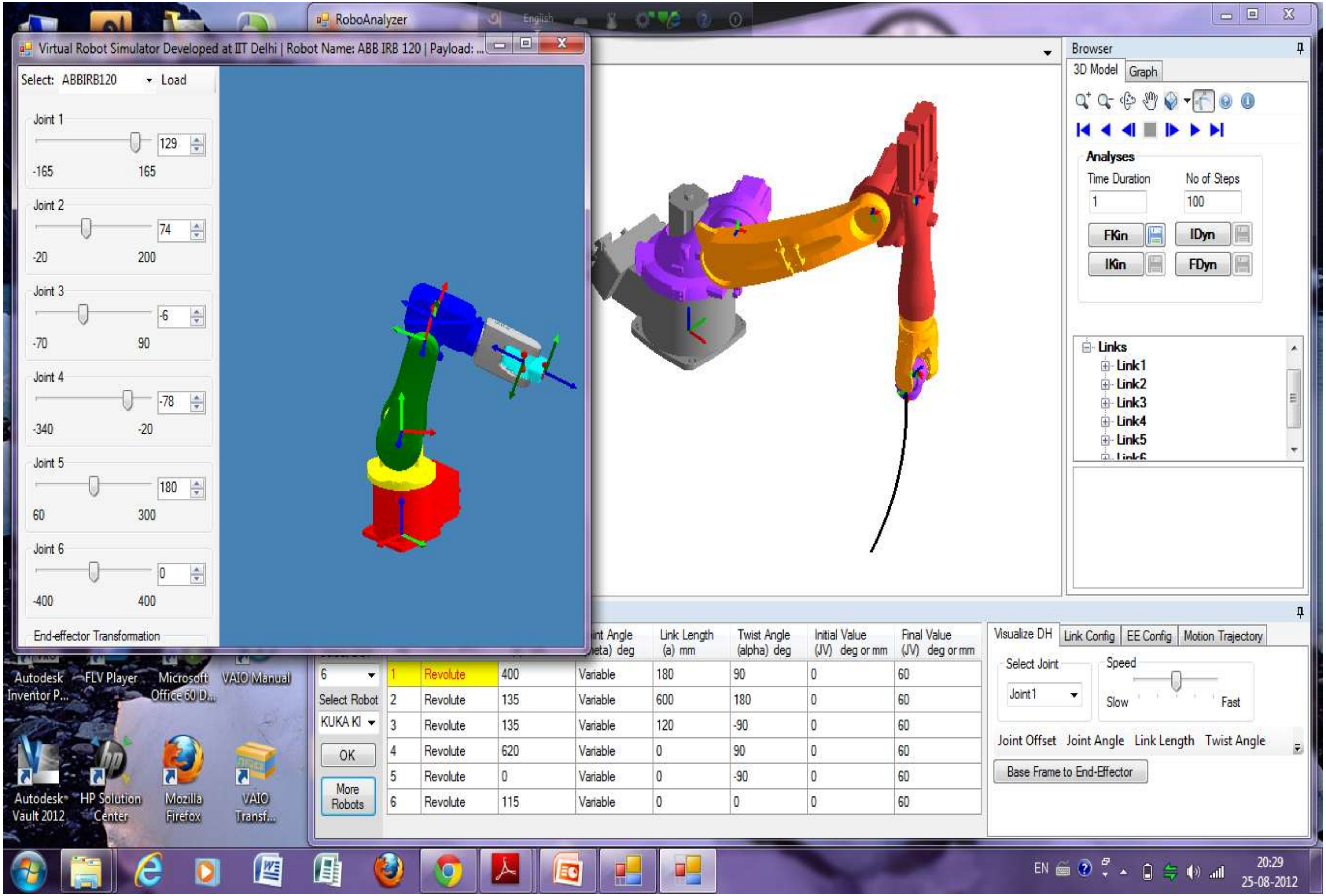
$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$$

⇓

Simulation

Integrate (say, numerically) to obtain **y**(t) using, e.g., Runge-Kutta method (ode45 of MATLAB)

Simulation using RA



Recursive Forward Dynamics

Equation of the motion

$$\boxed{\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \boldsymbol{\tau}} \quad \mathbf{UDU}^T\ddot{\boldsymbol{\theta}} = \boldsymbol{\varphi}, \text{ where } \mathbf{I} = \underbrace{\mathbf{UDU}^T}_{Analytically} \text{ and } \boldsymbol{\varphi} = \boldsymbol{\tau} - \mathbf{C}\dot{\boldsymbol{\theta}}$$

The joint accelerations are then solved as

$$\ddot{\boldsymbol{\theta}} = \mathbf{U}^{-T}\mathbf{D}^{-1}\mathbf{U}^{-1}\boldsymbol{\varphi}$$

Hence forward dynamics requires three steps

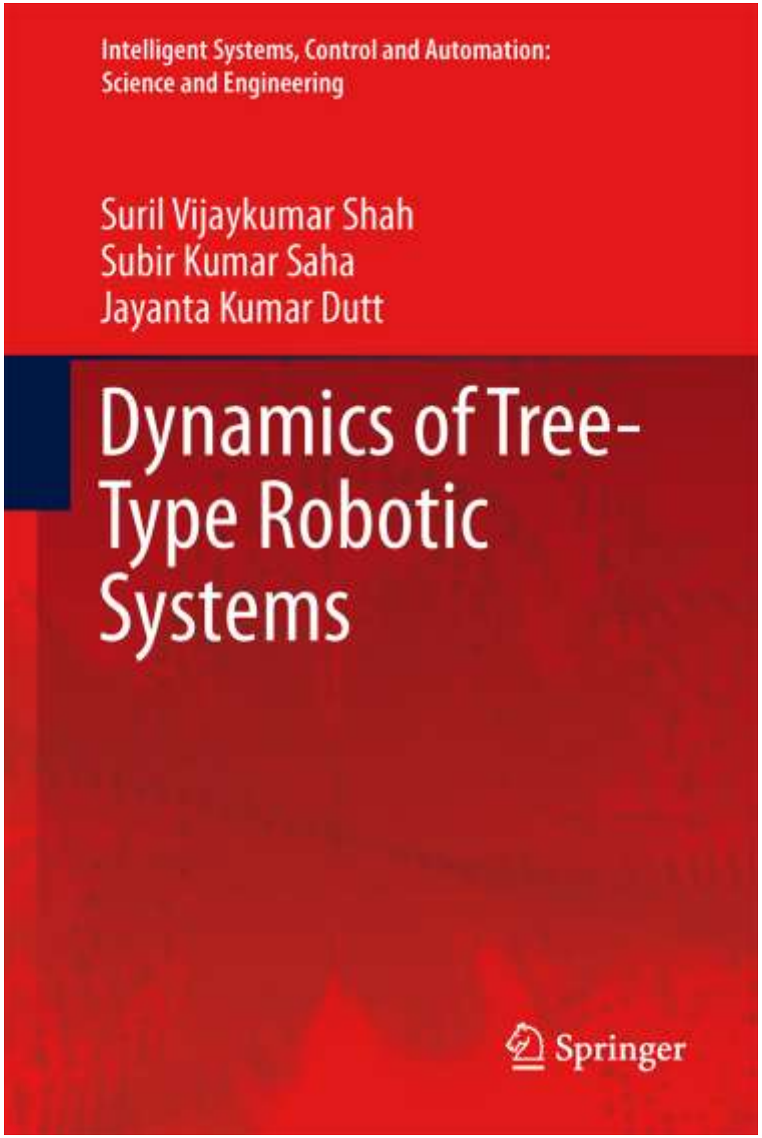
Step 1: Computation of $\boldsymbol{\phi}$

Step 2: $\mathbf{UDU}^T$ Decomposition

Step 3: Recursive computation of $\ddot{\boldsymbol{\theta}}$

Observations

- Derivations appear to be complex for the simpler manipulators
- It was for demonstration only
- The computations will be done algorithmically
- Due to recursive natures, calculations are fast
- The algorithm should be used for complex robotic systems like 6- or more-DOF robots



Lecture 3

Recursive Robot Dynamics

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Summary of Previous Lecture

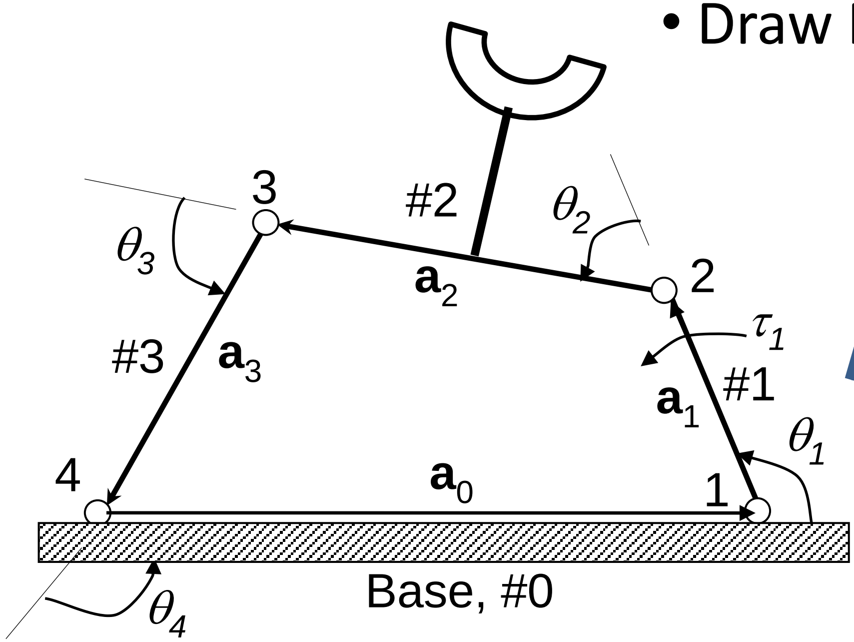
- Use of RoboAnalyzer (RA) for inverse dynamics of one-link arm
- GIM for 2-link manipulator
- Inverse dynamics of KUKA robot
- Forward dynamics and simulation
- Use of RA for simulation
- Some observations for using recursive dynamics

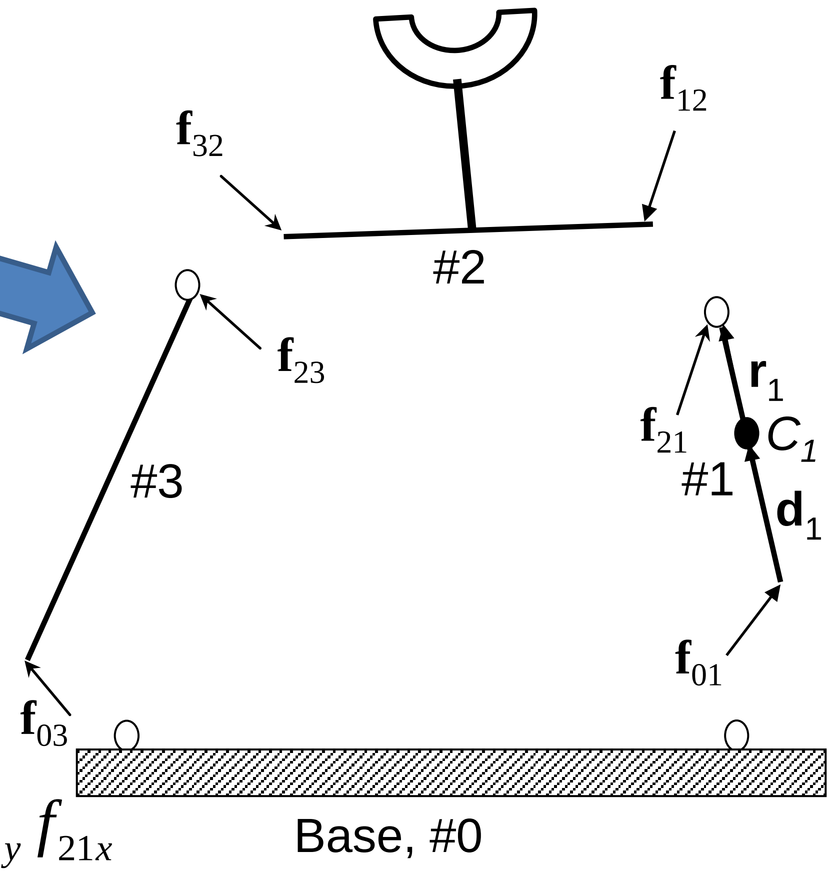
Parallel Manipulators

Stewart Platform

Four-bar Mechanism

- Separate all links
- Draw Free-body diagrams (FBD)





Equations of motion: 3 per link

$$I_1\ddot{\theta}_1 = \tau_1 - d_{1x} f_{01y} + d_{1y} f_{01x} + r_{1x} f_{21y} - r_{1y} f_{21x}$$

$$m_1\ddot{c}_{1x} = f_{01x} + f_{21x}\,;\; m_1\ddot{c}_{1y} = f_{01y} + f_{21y}$$

$$-f_{12x} \qquad\qquad -f_{12y}$$

9 equations

13 – 4 (3rd law) = 9 unknowns

$$
\begin{bmatrix}
1 & d_{1y} & -d_{1x} & r_{1y} & -r_{1x} & & & & \\
 & 1 & & -1 & & & 0's & & \\
 & & 1 & & -1 & & & & \\
 & & & d_{2y} & -d_{2x} & r_{2y} & -r_{2x} & & \\
 & & & 1 & & -1 & & & \\
 & & & & 1 & & -1 & & \\
 & & & & & d_{3y} & -d_{3x} & -r_{3y} & r_{3x} \\
 & & 0's & & & 1 & & -1 & \\
 & & & & & & 1 & & -1
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\ f_{01x} \\ f_{01y} \\ f_{12x} \\ f_{12x} \\ f_{23x} \\ f_{23y} \\ f_{03x} \\ f_{03y}
\end{bmatrix}
=
\begin{bmatrix}
I_1\ddot{\theta}_1 \\ m_1\ddot{c}_{1x} \\ m_1\ddot{c}_{1x} \\ I_2\ddot{\theta}_2 \\ m_2\ddot{c}_{2x} \\ m_2\ddot{c}_{2y} \\ I_3\ddot{\theta}_3 \\ m_3\ddot{c}_{3x} \\ m_3\ddot{c}_{3y}
\end{bmatrix}
$$

$\mathbf{A}: 9\times 9$ matrix $\qquad \mathrm{x}: 9\times 1 \quad \mathbf{b}: 9\times 1$

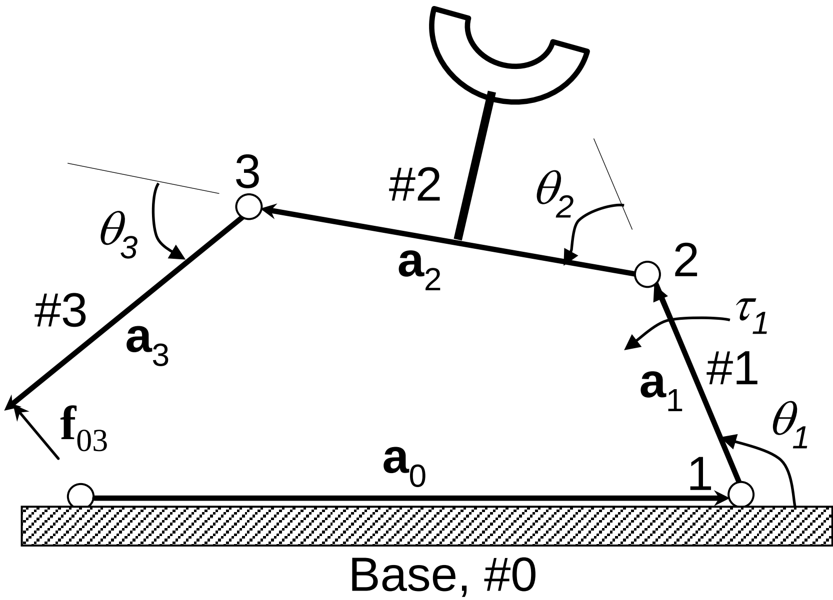
- Ghosh and Mallik, *Theory of Machines and Mechanisms*

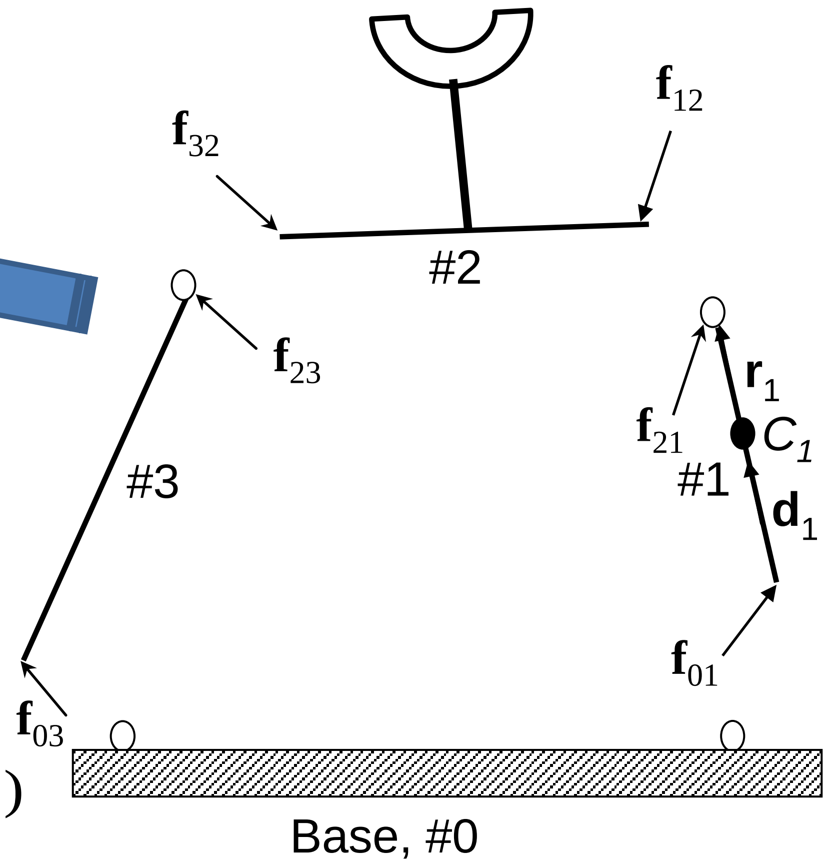
$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b} \Leftrightarrow \mathbf{L}\underbrace{\mathbf{Ux}}_{y}=\mathbf{b};\ \mathbf{Ly}=\mathbf{b}\ \text{(Forward)};\ \mathbf{Ux}=\mathbf{y}\ \text{(Backward)}$$

<u>Disadv.</u>: Need to calculate even the reactions for inv. dyn.

Three-link Serial with $\mathbf{f}_{03}$ as External

- Join first three links form





Equations of motion

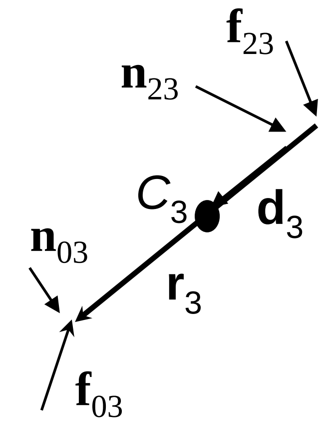
$$\mathbf{N}_d^T\mathbf{N}_l^T(\mathbf{M}\dot{\mathbf{t}}+\mathbf{W}\mathbf{M}\mathbf{t})=\mathbf{N}_d^T\mathbf{N}_l^T(\mathbf{w}^E+\mathbf{w}^C)$$

$$\underset{3\times3}{\mathbf{I}}\,\ddot{\boldsymbol{\theta}}+\mathbf{C}\dot{\boldsymbol{\theta}}=\boldsymbol{\tau}^E+\underset{\mathbf{J}^T\mathbf{f}_{03}}{\boldsymbol{\tau}^C}\quad : 3 \text{ eqs.}$$

$$\underset{\mathbf{0}}{\dot{\mathbf{a}}_0}+\dot{\mathbf{a}}_1+\dot{\mathbf{a}}_2+\dot{\mathbf{a}}_3=\mathbf{0}\rightarrow\underset{2\times3}{\mathbf{J}}\,\dot{\boldsymbol{\theta}}=\mathbf{0}$$

Proof of $\tau^C = J^T f_{03}$

- The DeNOC matrices for 3-link serial manipulator



$$\underbrace{\begin{bmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \mathbf{t}_3\end{bmatrix}}_{18\times 1} = \underbrace{\begin{bmatrix}\mathbf{1} & & 0's\\ \mathbf{B}_{21} & \mathbf{1} & \\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{1}\end{bmatrix}}_{\mathbf{N}_l:18\times 18}\underbrace{\begin{bmatrix}\mathbf{p}_1 & & 0's\\ & \mathbf{p}_2 & \\ 0's & & \mathbf{p}_3\end{bmatrix}}_{\mathbf{N}_d:18\times 3}\underbrace{\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}}_{3\times 1}$$

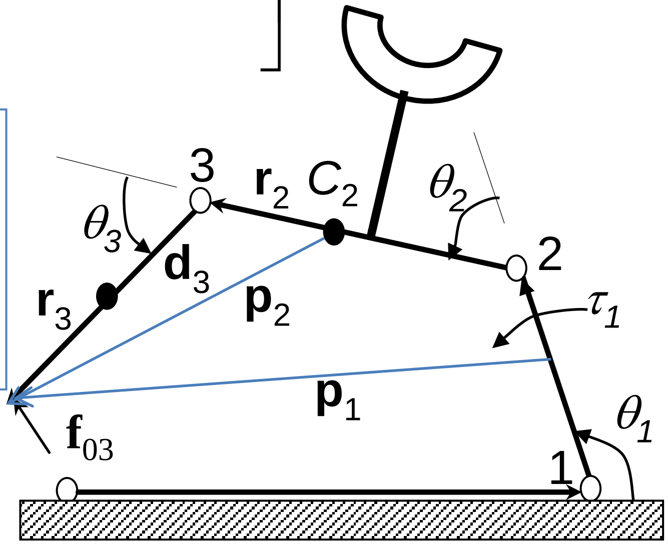
$$\mathbf{N}_l^T\mathbf{w}^C = \begin{bmatrix}\mathbf{1} & \mathbf{B}_{21}^T & \mathbf{B}_{31}^T\\ & \mathbf{1} & \mathbf{B}_{32}^T\\ 0's & & \mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{w}_1^C\\ \mathbf{w}_2^C\\ \mathbf{w}_3^C\end{bmatrix} = \begin{bmatrix}\mathbf{w}_1^C + \underbrace{\mathbf{B}_{21}^T(\mathbf{w}_2^C + \mathbf{B}_{32}^T\mathbf{w}_3^C)}_{\because \mathbf{B}_{32}\mathbf{B}_{21}=\mathbf{B}_{31}\Rightarrow \mathbf{B}_{21}^T\mathbf{B}_{32}^T=\mathbf{B}_{31}^T}\\ \underbrace{\mathbf{w}_2^C + \mathbf{B}_{32}^T\mathbf{w}_3^C}_{\tilde{\mathbf{w}}_2^C}\\ \mathbf{w}_3^C\end{bmatrix}$$

$$\underset{6\times 6}{\mathbf{B}_{32}^T} = \begin{bmatrix}\mathbf{1} & (\mathbf{r}_2+\mathbf{d}_3)\times\mathbf{1}\\ 0's & \mathbf{1}\end{bmatrix} \qquad \mathbf{w}_3^C = \begin{bmatrix}\mathbf{n}_{03}+\mathbf{n}_{23}-\mathbf{d}_3\times\mathbf{f}_{23}+\mathbf{r}_3\times\mathbf{f}_{03}\\ \mathbf{f}_{03}+\mathbf{f}_{23}\end{bmatrix}$$

$$\underset{6\times 1}{\tilde{\mathbf{w}}_2^C} = \mathbf{w}_2^C + \mathbf{B}_{32}^T \mathbf{w}_3^C = \begin{bmatrix} \cancel{\mathbf{n}_{32}} + \mathbf{n}_{12} - \mathbf{d}_2 \times \mathbf{f}_{12} + \cancel{\mathbf{r}_2 \times \mathbf{f}_{32}} \\ -\mathbf{n}_{23} \\ \cancel{\mathbf{f}_{32}} + \mathbf{f}_{12} \\ -\mathbf{f}_{23} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{n}_{03} + \cancel{\mathbf{n}_{23}} - \cancel{\mathbf{d}_3 \times \mathbf{f}_{23}} + \mathbf{r}_3 \times \mathbf{f}_{03} + (\mathbf{r}_2 + \mathbf{d}_3) \times (\mathbf{f}_{03} + \mathbf{f}_{23}) \\ \mathbf{f}_{03} + \cancel{\mathbf{f}_{23}} \end{bmatrix}$$

$$\tilde{\mathbf{w}}_2^C = \begin{bmatrix} \mathbf{n}_{03} + \mathbf{n}_{12} + \underbrace{(\mathbf{r}_2 + \mathbf{d}_3 + \mathbf{r}_3)}_{\mathbf{p}_2} \times \mathbf{f}_{03} - \mathbf{d}_2 \times \mathbf{f}_{12} \\ \mathbf{f}_{03} + \mathbf{f}_{12} \end{bmatrix}$$



$$\tilde{\mathbf{w}}_1^C = \begin{bmatrix} \mathbf{n}_{03} + \mathbf{n}_{01} + \mathbf{p}_1 \times \mathbf{f}_{03} - \mathbf{d}_1 \times \mathbf{f}_{01} \\ \mathbf{f}_{03} + \mathbf{f}_{01} \end{bmatrix}$$

Constraint Torque

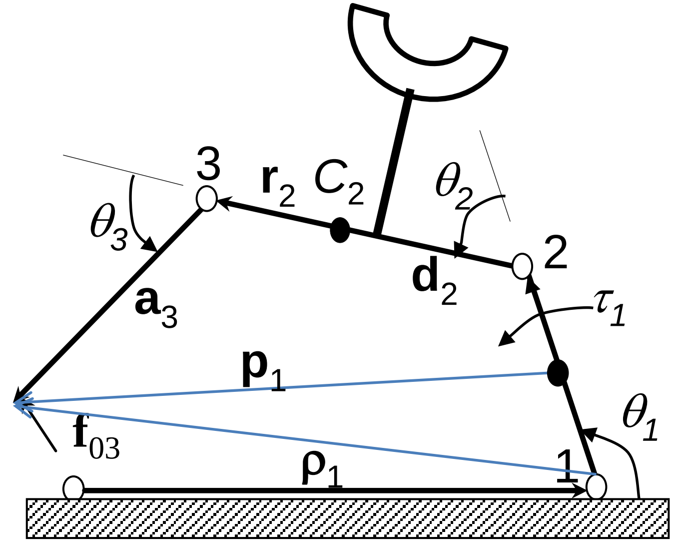
$$\mathbf{N}_d^T \tilde{\mathbf{w}}^C = \begin{bmatrix} \mathbf{p}_1^T & & 0's \\ & \mathbf{p}_2^T & \\ 0's & & \mathbf{p}_3^T \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_1^C \\ \tilde{\mathbf{w}}_2^C \\ \mathbf{w}_3^C \end{bmatrix} \qquad \mathbf{p}_i\,(i=1,2,3) = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_i \times \mathbf{d}_i \end{bmatrix}$$

$$\underset{Scalar}{\tau_1^C} = \mathbf{p}_1^T \tilde{\mathbf{w}}_1^C = \begin{bmatrix} \mathbf{e}_1^T & (\mathbf{e}_1 \times \mathbf{d}_1)^T \end{bmatrix} \begin{bmatrix} \mathbf{n}_{03} + \mathbf{n}_{01} + \mathbf{p}_1 \times \mathbf{f}_{03} - \mathbf{d}_1 \times \mathbf{f}_{01} \\ \mathbf{f}_{03} + \mathbf{f}_{01} \end{bmatrix}$$

$$= \mathbf{e}_1^T (\mathbf{n}_{03} + \mathbf{n}_{01} + \mathbf{p}_1 \times \mathbf{f}_{03} - \mathbf{d}_1 \times \mathbf{f}_{01})$$

$$+ (\mathbf{e}_1 \times \mathbf{d}_1)^T (\mathbf{f}_{03} + \mathbf{f}_{01})$$

$$= \mathbf{e}_1^T (\boldsymbol{\rho}_1 \times \mathbf{f}_{03}) = (\mathbf{e}_1 \times \boldsymbol{\rho}_1)^T \mathbf{f}_{03}$$



$$\tau_2^C = \mathbf{p}_2^T \tilde{\mathbf{w}}_2^C = \mathbf{e}_2^T (\boldsymbol{\rho}_2 \times \mathbf{f}_{03}) = (\mathbf{e}_2 \times \boldsymbol{\rho}_2)^T \mathbf{f}_{03}$$

$$\tau_3^C = \mathbf{p}_3^T \mathbf{w}_3^C = \mathbf{e}_3^T (\mathbf{a}_3 \times \mathbf{f}_{03}) = (\mathbf{e}_2 \times \mathbf{a}_3)^T \mathbf{f}_{03}$$

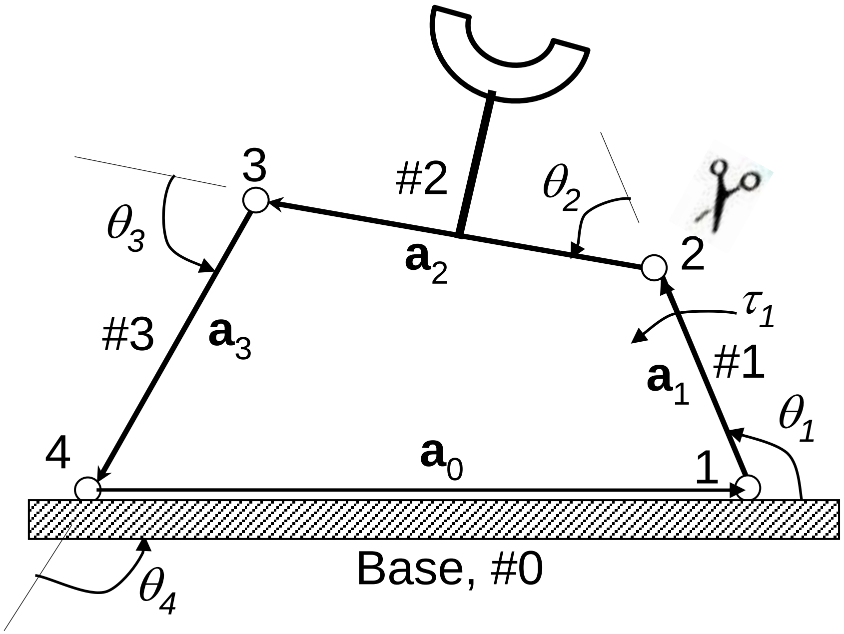
Equations of Motion

$$\underbrace{\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}}}_{\boldsymbol{\tau}^*:known} = \underbrace{\boldsymbol{\tau}^E}_{[\tau_1,0,0]^T} + \underbrace{\boldsymbol{\tau}^C}_{\mathbf{J}^T\mathbf{f}_{03}} \qquad \mathbf{J} = \begin{bmatrix} \mathbf{e}_1 \times \boldsymbol{\rho}_1 & \mathbf{e}_2 \times \boldsymbol{\rho}_2 & \mathbf{e}_3 \times \underset{\mathbf{a}_3}{\boldsymbol{\rho}_3} \end{bmatrix}$$

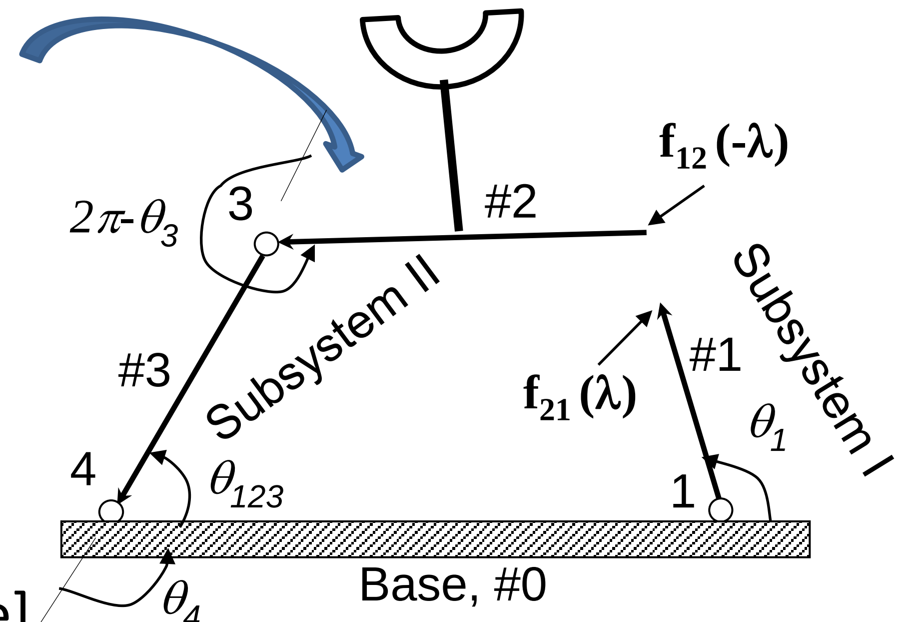
$$\underbrace{\begin{bmatrix} \tau_1^* \\ \tau_2^* \\ \tau_3^* \end{bmatrix}}_{\mathbf{b}:3\times1} = \underbrace{\begin{bmatrix} 1 & -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ 0 & -a_2 s_{12} - a_3 s_{123} & a_2 c_{12} + a_3 c_{123} \\ 0 & -a_3 s_{123} & a_3 c_{123} \end{bmatrix}}_{\mathbf{A}:3\times3} \underbrace{\begin{bmatrix} \tau_1 \\ f_{03x} \\ f_{03y} \end{bmatrix}}_{\mathbf{x}:3\times1}$$

<u>Adv.</u>: Reduced size of 3×3 (instead of 9×9) for inverse dynamics

Subsystem Recursive Method



- Cut-open in a suitable location
- Make serial systems
- Apply serial-chain methods



Subsystem equations

$$\mathbf{I}^{I}\ddot{\boldsymbol{\theta}}^{I}+\mathbf{C}^{I}\dot{\boldsymbol{\theta}}^{I}=\boldsymbol{\tau}^{I}+(\boldsymbol{\tau}^{C})^{I}$$

: 1 eq., 3 (τ, λ_x, λ_y)? [1R: done]

$$\mathbf{I}^{II}\ddot{\boldsymbol{\theta}}^{II}+\mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II}=\boldsymbol{\tau}^{II}+(\boldsymbol{\tau}^{C})^{II}$$

: 2 eqs., 2 (λ_x, λ_y)? [2R: done]

$$\mathbf{a}_1=-\mathbf{a}_0-\mathbf{a}_2-\mathbf{a}_3$$

$$\dot{\mathbf{a}}_1 = -\dot{\mathbf{a}}_2 - \dot{\mathbf{a}}_3 \Rightarrow \mathbf{J}^I \dot{\theta}^I = \mathbf{J}^{II} \dot{\boldsymbol{\theta}}^{II}$$

$$\dot{\theta}^I \equiv \dot{\theta}_1; \dot{\boldsymbol{\theta}}^{II} \equiv \begin{bmatrix} \dot{\theta}_1^{II} \\ \dot{\theta}_{123} \\ \dot{\theta}_2^{II} \\ \dot{\theta}_3 \end{bmatrix}$$

$$\underset{2\times1}{\mathbf{J}^I} \equiv \underbrace{\begin{bmatrix} -a_1 s_1 \\ a_1 c_1 \end{bmatrix}}_{\mathbf{A}_{en}^I \mathbf{N}_{ln}^I \mathbf{N}_d^I}; \underset{2\times2}{\mathbf{J}^{II}} \equiv \underbrace{\begin{bmatrix} a_3 s_{123} + a_2 s_{12} & -a_2 s_{12} \\ -a_3 c_{123} - a_2 c_{12} & a_2 c_{12} \end{bmatrix}}_{\mathbf{A}_{en}^{II} \mathbf{N}_{ln}^{II} \mathbf{N}_d^{II}}$$

System + Motion ↓

Subsystem II: $\mathbf{I}^{II} \ddot{\boldsymbol{\theta}}^{II} + \mathbf{C}^{II} \dot{\boldsymbol{\theta}}^{II} = \underbrace{\boldsymbol{\tau}^{II}}_{[0,0]^T} + \underbrace{(\boldsymbol{\tau}^C)^{II}}_{-(\mathbf{J}^{II})^T \boldsymbol{\lambda}}$ → Solve λ

Subsystem I: $I^I \ddot{\theta}^I + C^I \dot{\theta}^I = \tau^I + \underbrace{(\tau^C)^I}_{(\mathbf{J}^1)^T \boldsymbol{\lambda}}$ ← (from Solve λ)

↑ System + Motion

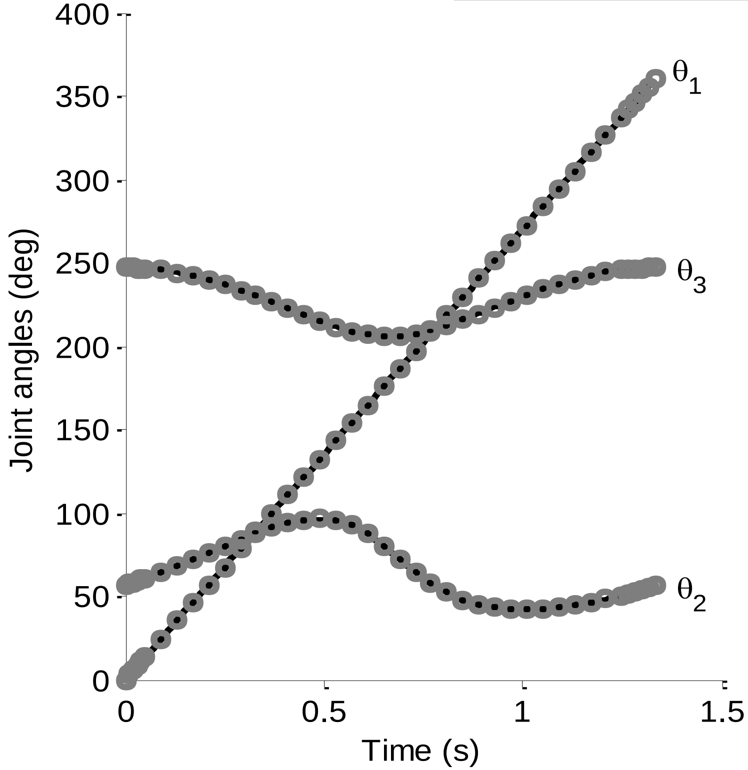
↓ Driving torque, τ_1

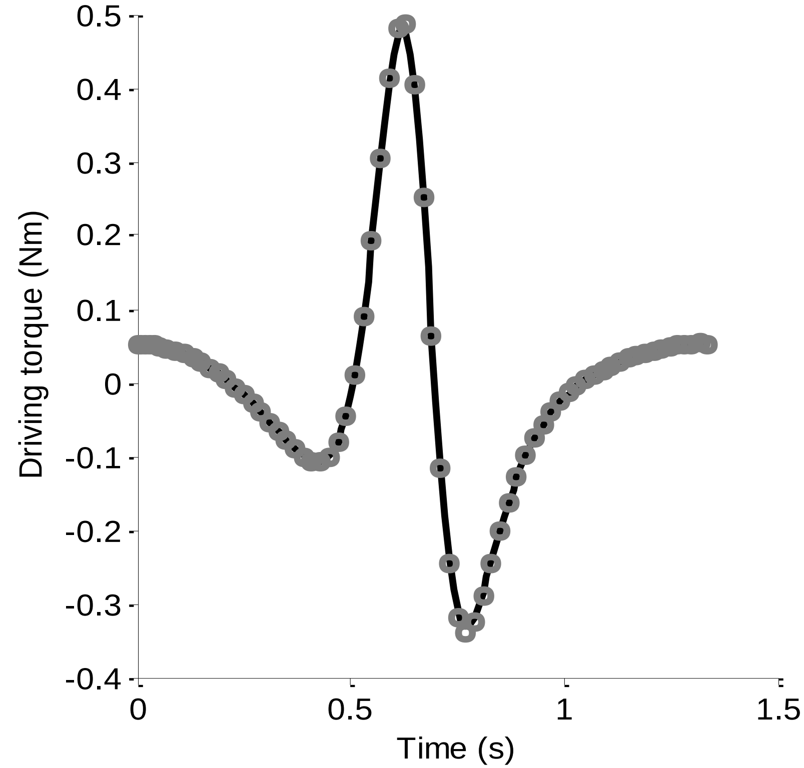
Adv.: Subsystem recursion.; Maximum size: 2×2 (not 9×9 or 3×3); Can use existing serial-chain dyn. algo.

Using ReDySim
(MATLAB-based)

Free: http://www.redysim.co.nr/download

Sub-system (link)	Mass (Kg)	Length (m)
I(#1)	1.5	0.038
II(#2)	5	0.2304
II(#3)	3	0.1152





Lecture 4

Recursive Robot Dynamics

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Summary of Previous Lecture

- Dynamics (classical way) of 4-bar mechanism using FBD
- Cut-open 3+0 constrained dynamics of 4-bar
- Cut-open 2+1 constrained dynamics of 4-bar
- Inverse dynamics using ReDySim

Forward Dynamics of 4-bar Mechanism

Constrained equations of motion:

$$I^I \ddot{\theta}^I + C^I \dot{\theta}^I = \underbrace{\tau^I}_{\tau_1} + \underbrace{(\tau^C)^I}_{(\mathbf{J}^1)^T \boldsymbol{\lambda}} \qquad \mathbf{I}^{II}\ddot{\boldsymbol{\theta}}^{II} + \mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} = \underbrace{\boldsymbol{\tau}^{II}}_{[0,0]^T} + \underbrace{(\boldsymbol{\tau}^C)^{II}}_{-(\mathbf{J}^{II})^T \boldsymbol{\lambda}}$$

+

Velocity constraints:

$$\begin{bmatrix} \mathbf{J}^I & \text{-}\mathbf{J}^{II} \end{bmatrix} \begin{bmatrix} \dot{\theta}^I \\ \dot{\boldsymbol{\theta}}^{II} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}$$

= DAE [Diff. Algeb. Eqn.) formulation: System appr.]

$$\underbrace{\begin{bmatrix} I^I & \mathbf{O} & (\mathbf{J}^I)^T \\ 0 & \mathbf{I}^{II} & \text{-}(\mathbf{J}^{II})^T \\ \mathbf{J}^I & \text{-}\mathbf{J}^{II} & \mathbf{O} \end{bmatrix}}_{\mathbf{A}:5\times 5} \underbrace{\begin{bmatrix} \ddot{\theta}^I \\ \ddot{\boldsymbol{\theta}}^{II} \\ \boldsymbol{\lambda} \end{bmatrix}}_{\mathbf{x}:5\times 1} = \underbrace{\begin{bmatrix} \tau_1 - C^I \dot{\theta}^I \\ \text{-}\mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} \\ \text{-}\dot{\mathbf{J}}^I \dot{\theta}^I + \dot{\mathbf{J}}^{II}\dot{\boldsymbol{\theta}}^{II} \end{bmatrix}}_{\mathbf{b}:5\times 1}$$

→

$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$

+

$\int$ (jt. accn.) → jt. vel. & pos.

Using ReDySim

System-level Lagrange Multiplier Method

Constrained equations of motion:

$$I^I\ddot{\theta}^I + C^I\dot{\theta}^I = \underbrace{\tau^I}_{\tau_1} + \underbrace{(\tau^C)^I}_{(\mathbf{J}^1)^T\boldsymbol{\lambda}}$$

$$\mathbf{I}^{II}\ddot{\boldsymbol{\theta}}^{II} + \mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} = \underbrace{\boldsymbol{\tau}^{II}}_{[0,0]^T} + \underbrace{(\boldsymbol{\tau}^C)^{II}}_{-(\mathbf{J}^{II})^T\boldsymbol{\lambda}}$$

$$\underbrace{\begin{bmatrix} I^I & \mathbf{O} \\ 0 & \mathbf{I}^{II} \end{bmatrix}}_{\mathbf{I}:3\times3}\underbrace{\begin{bmatrix} \ddot{\theta}^I \\ \ddot{\boldsymbol{\theta}}^{II} \end{bmatrix}}_{\ddot{\boldsymbol{\theta}}:3\times1} = \underbrace{\begin{bmatrix} \tau^I\text{-}C^I\dot{\theta}^I \\ \boldsymbol{\tau}^{II}\text{-}\mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} \end{bmatrix}}_{\boldsymbol{\phi}_1:3\times1} + \underbrace{\begin{bmatrix} (\mathbf{J}^I)^T \\ -(\mathbf{J}^{II})^T \end{bmatrix}}_{\mathbf{J}^T:3\times2}\boldsymbol{\lambda}$$

Velocity constraints:

$$\underbrace{\begin{bmatrix} \mathbf{J}^I & \text{-}\mathbf{J}^{II} \end{bmatrix}}_{\mathbf{J}}\underbrace{\begin{bmatrix} \ddot{\theta}^I \\ \ddot{\boldsymbol{\theta}}^{II} \end{bmatrix}}_{\ddot{\boldsymbol{\theta}}} = \underbrace{\text{-}\dot{\mathbf{J}}^I\dot{\theta}^I + \dot{\mathbf{J}}^{II}\dot{\boldsymbol{\theta}}^{II}}_{\boldsymbol{\phi}_2}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{J}^T \\ \mathbf{J} & \mathbf{O} \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{\theta}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix}$$

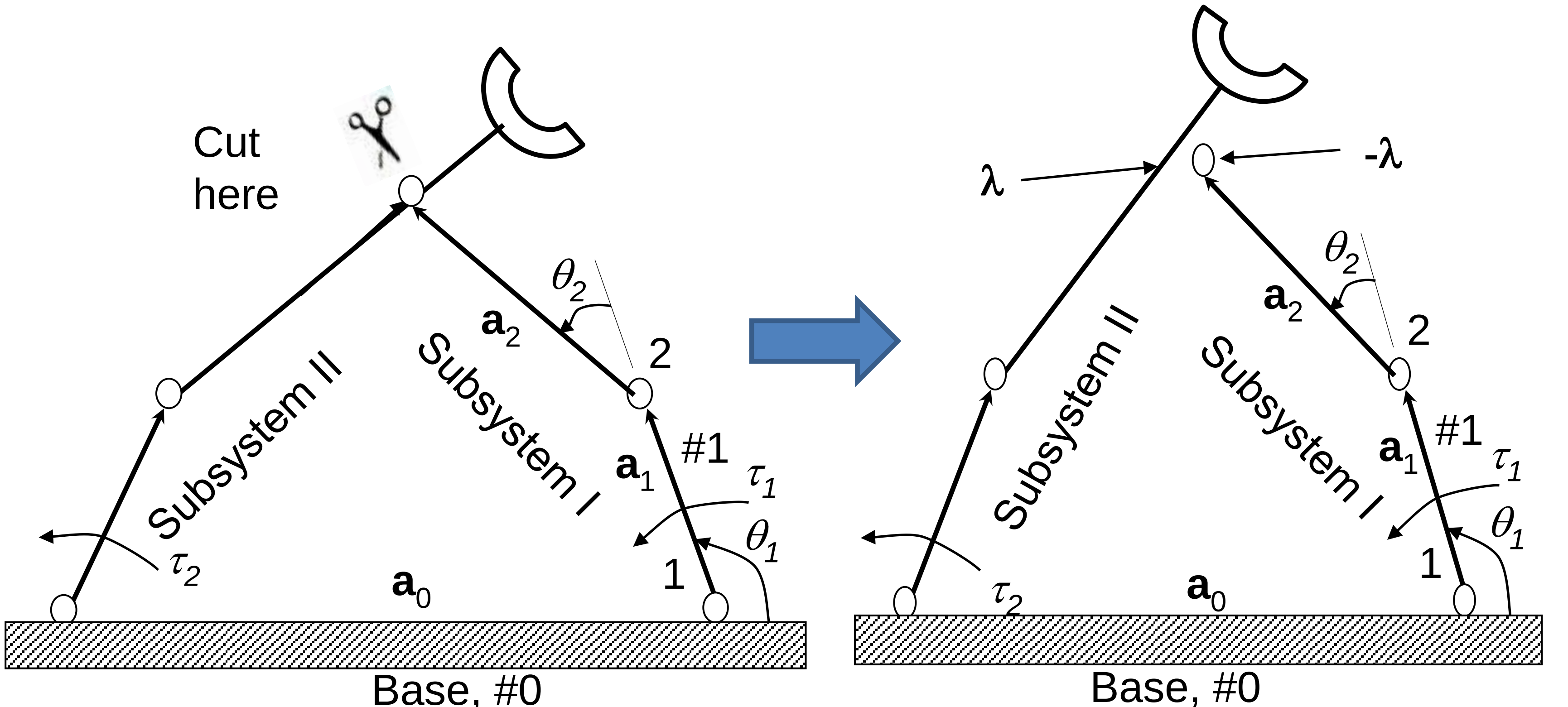
$$\boldsymbol{\lambda} = (\underbrace{\mathbf{J}\mathbf{I}^{-1}\mathbf{J}^T}_{\bar{\mathbf{I}}:2\times2})^{-1}(\mathbf{J}\mathbf{I}^{-1}\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{I}^{-1}(\boldsymbol{\phi}_1 - \mathbf{J}^T\boldsymbol{\lambda})$$

jt. vel. & pos. ←∫ (jt. accn.)

Adv.: Inversions of smaller (3×3 and 2×2) matrices

Five-bar 2-DOF Manipulator



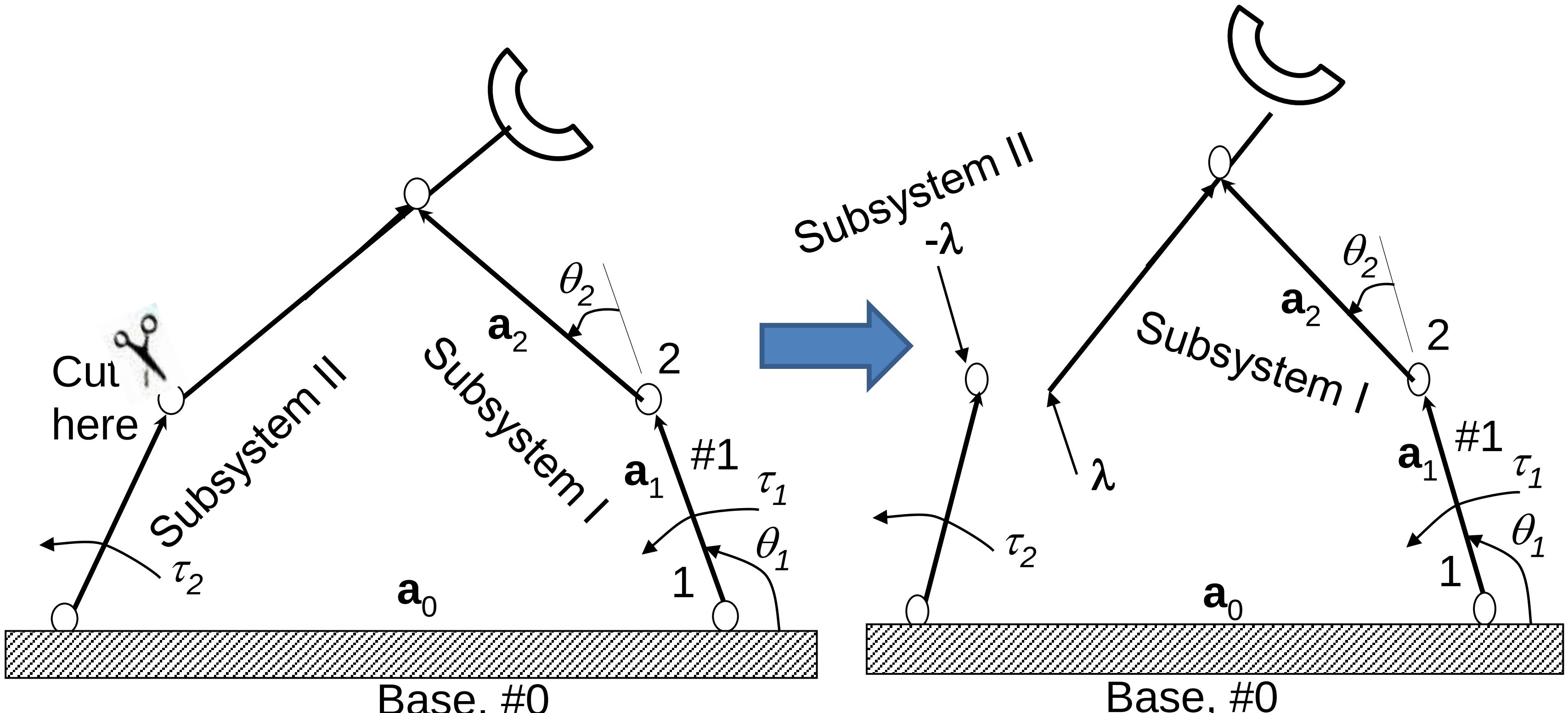
Subsystem equations

$$\mathbf{I}^I\ddot{\boldsymbol{\theta}}^I + \mathbf{C}^I\dot{\boldsymbol{\theta}}^I = \boldsymbol{\tau}^I + (\boldsymbol{\tau}^\lambda)^I$$: 2 eqs., 3 (τ_1, λ_x, λ_y)? [2R done]

$$\mathbf{I}^{II}\ddot{\boldsymbol{\theta}}^{II} + \mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} = \boldsymbol{\tau}^{II} + (\boldsymbol{\tau}^\lambda)^{II}$$: 2 eqs., 1 (τ_2)? [2R done]

Symmetric; Need to combine: 4 eqs., 4 (τ_1, τ_2, λ_x, λ_y)?

Subsystem Recursion for 5-bar



Subsystem equations

$$\mathbf{I}^I\ddot{\boldsymbol{\theta}}^I + \mathbf{C}^I\dot{\boldsymbol{\theta}}^I = \boldsymbol{\tau}^I + (\boldsymbol{\tau}^\lambda)^I$$

: 3 eqs., 3 (τ_1, λ_x, λ_y)? [3R done]

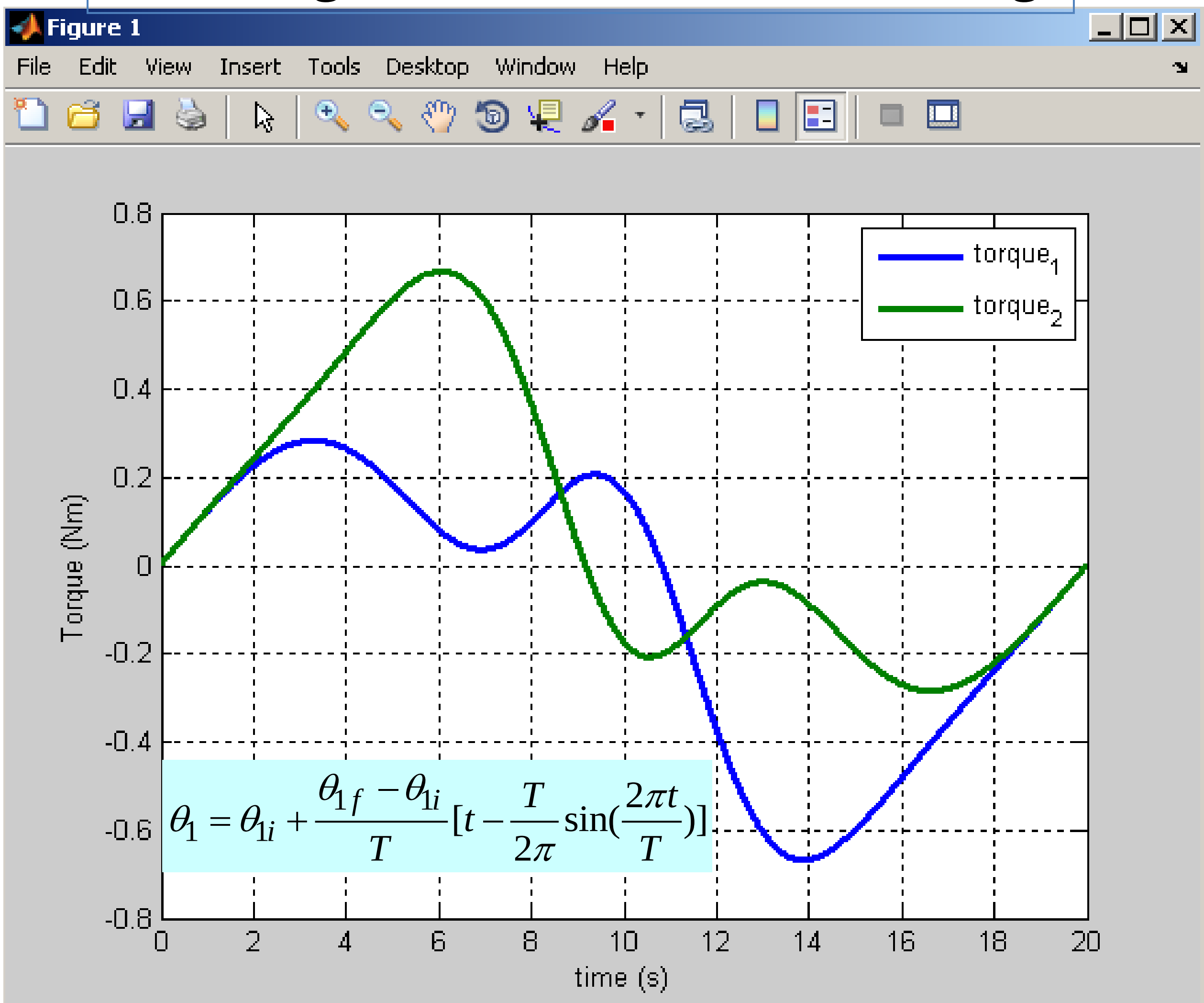
$$\mathbf{I}^{II}\ddot{\boldsymbol{\theta}}^{II} + \mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} = \boldsymbol{\tau}^{II} + (\boldsymbol{\tau}^\lambda)^{II}$$

: 1 eq., 1 (τ_2)? [1R done]

No need to combine: Solved

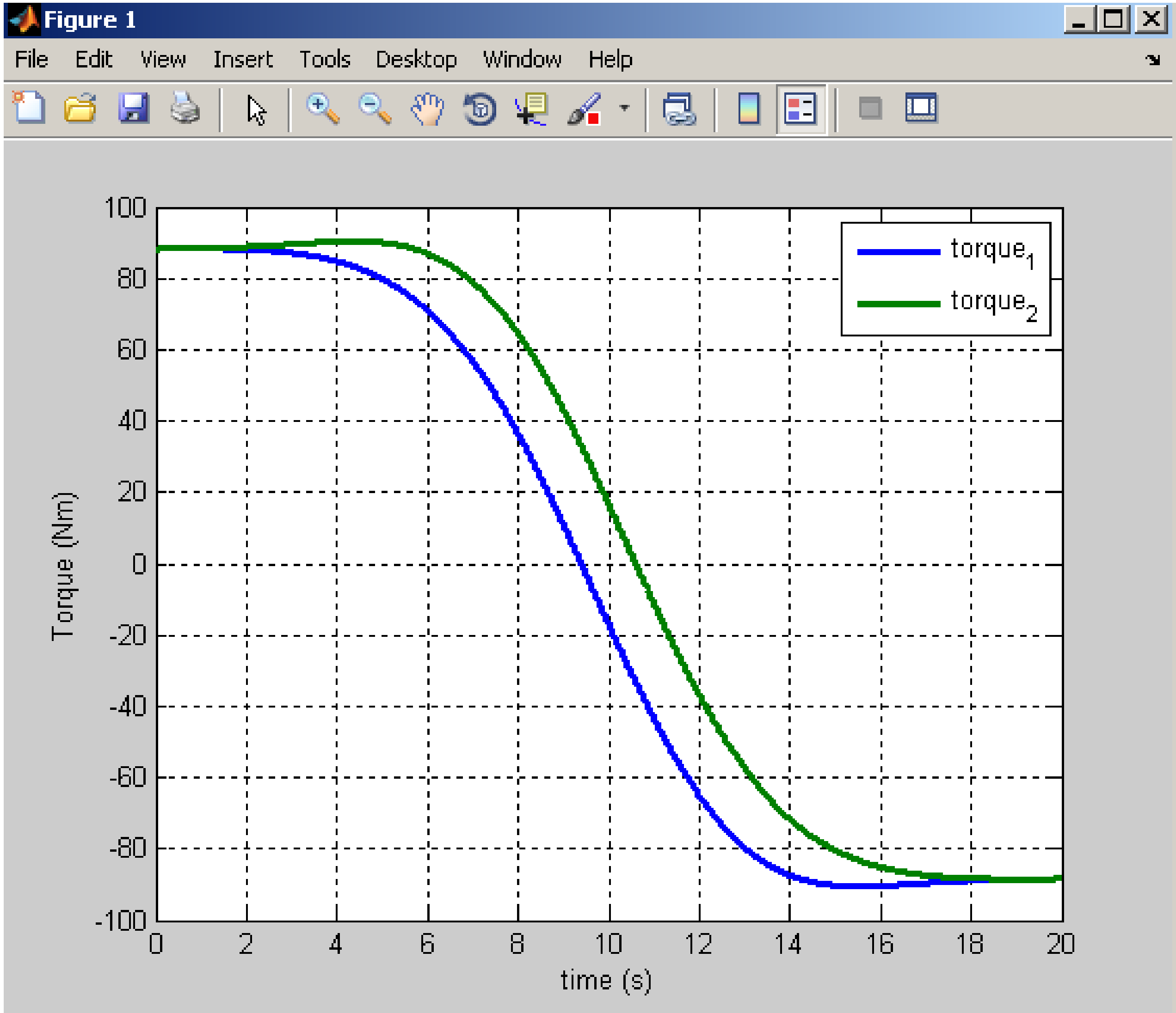
Inverse Dynamics (w/o Gravity)

Link lengths: 1 m; Mass of links: 6 kg

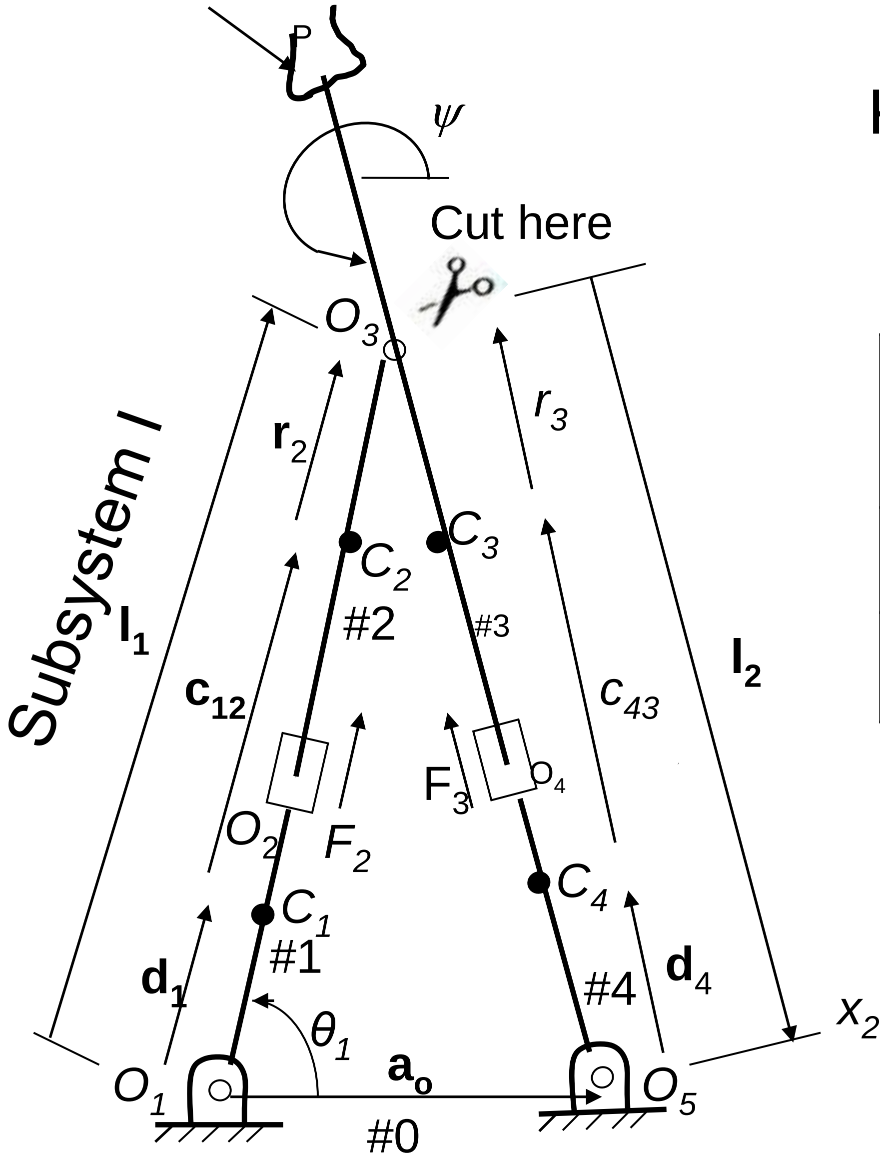


$$\theta_1 = \theta_{1i} + \frac{\theta_{1f} - \theta_{1i}}{T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right]$$

Inverse Dynamics (w/Gravity)



Two-DOF Parallel Manipulator



Kinematic constraints

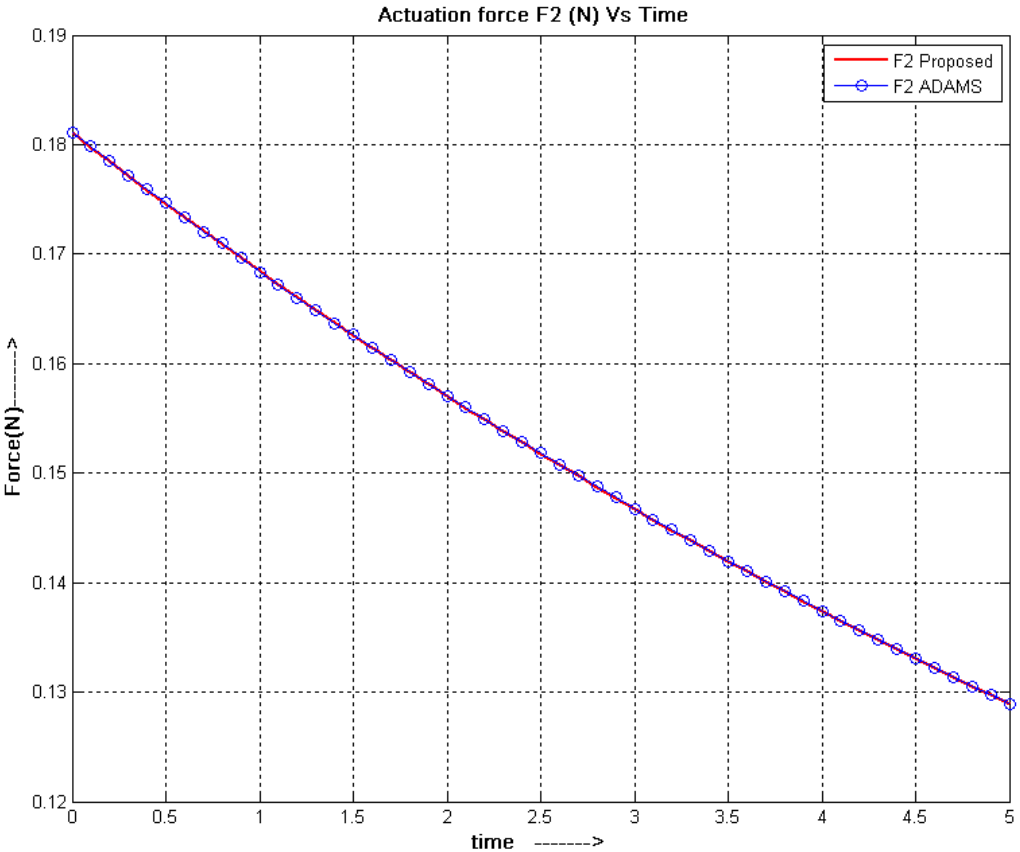
$$\dot{\mathbf{l}}_1 + \dot{\mathbf{l}}_2 = \dot{\mathbf{a}}_0 = \mathbf{0}$$

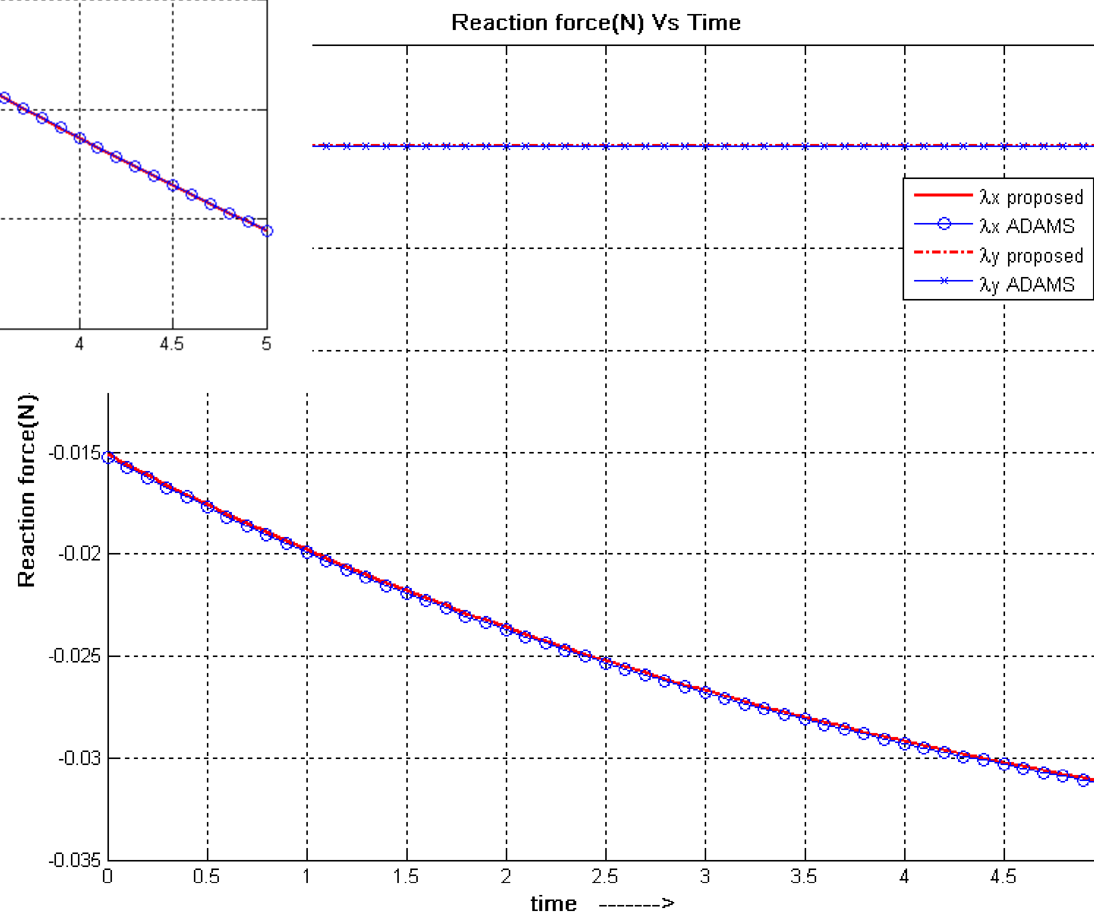
Link	b_i (m)	θ_i (m)	a_i (m)	α_i (m)
1	0	θ_1	0	$\pi/2$
2	b_2 [JV]	0	0	0

Link	m_i (kg)	$r_{i,x}, r_{i,y}$ (m)	$r_{i,z}$	$I_{i,xx}, I_{i,yy}$ (kg-m²)
1,2	655	0	.625	399

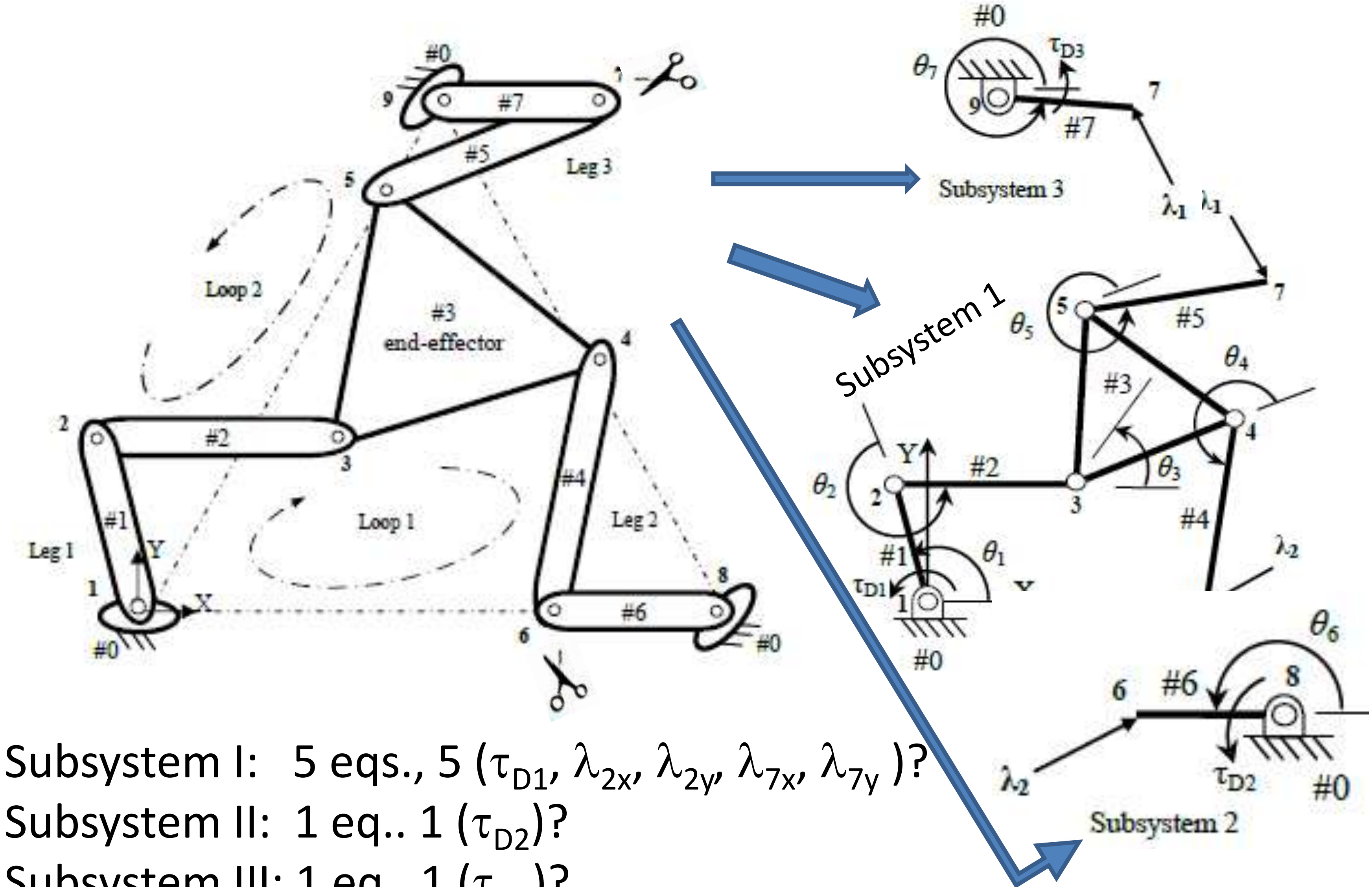
Sliding velocity (constant): 0.1 m/sec.

Two RP serial manipulators; Combined: 4 eqs., 4 (f_1, f_2, λ_x, λ_y)?


Actuation force F2 (N) Vs Time
F2 Proposed
F2 ADAMS
Force(N)-------->
time -------->
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Reaction force(N) Vs Time
λx proposed
λx ADAMS
λy proposed
λy ADAMS
Reaction force(N)
time -------->
-0.015
-0.02
-0.025
-0.03
-0.035
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Three-DOF RRR Parallel Robot



Subsystem I: 5 eqs., 5 (τ_{D1}, λ_{2x}, λ_{2y}, λ_{7x}, λ_{7y})?

Subsystem II: 1 eq.. 1 (τ_{D2})?

Subsystem III: 1 eq.. 1 (τ_{D3})?

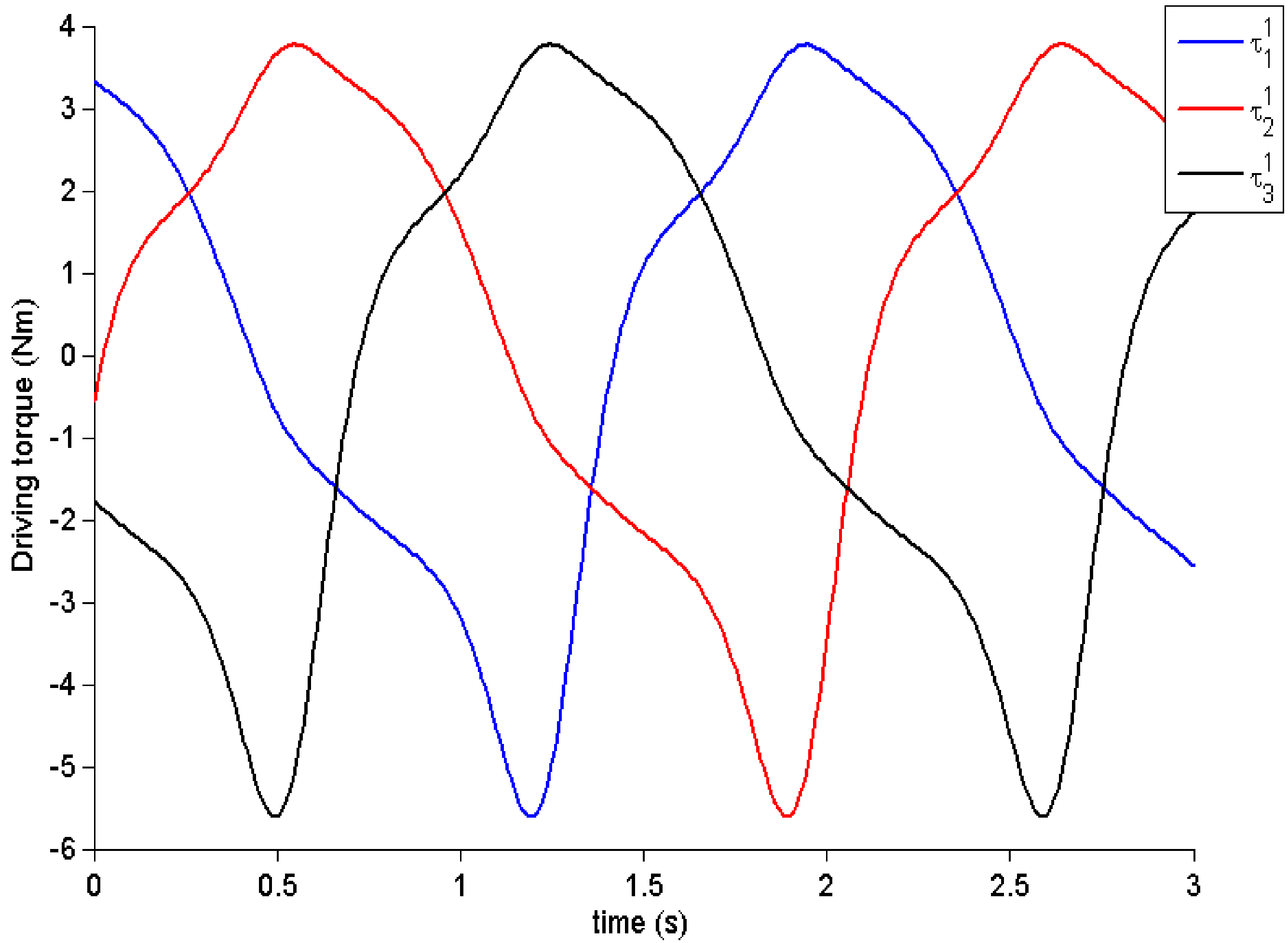
Inverse Dynamics of 3-DOF RRR Robot

	3-RRR parallel manipulator		
Sub-system	Link #	Length (m)	Mass (kg)
I	1	0.4	3
	2, 4, 5	0.6	4
	3	0.4*	8
II	6	0.4	3
III	7	0.4	3

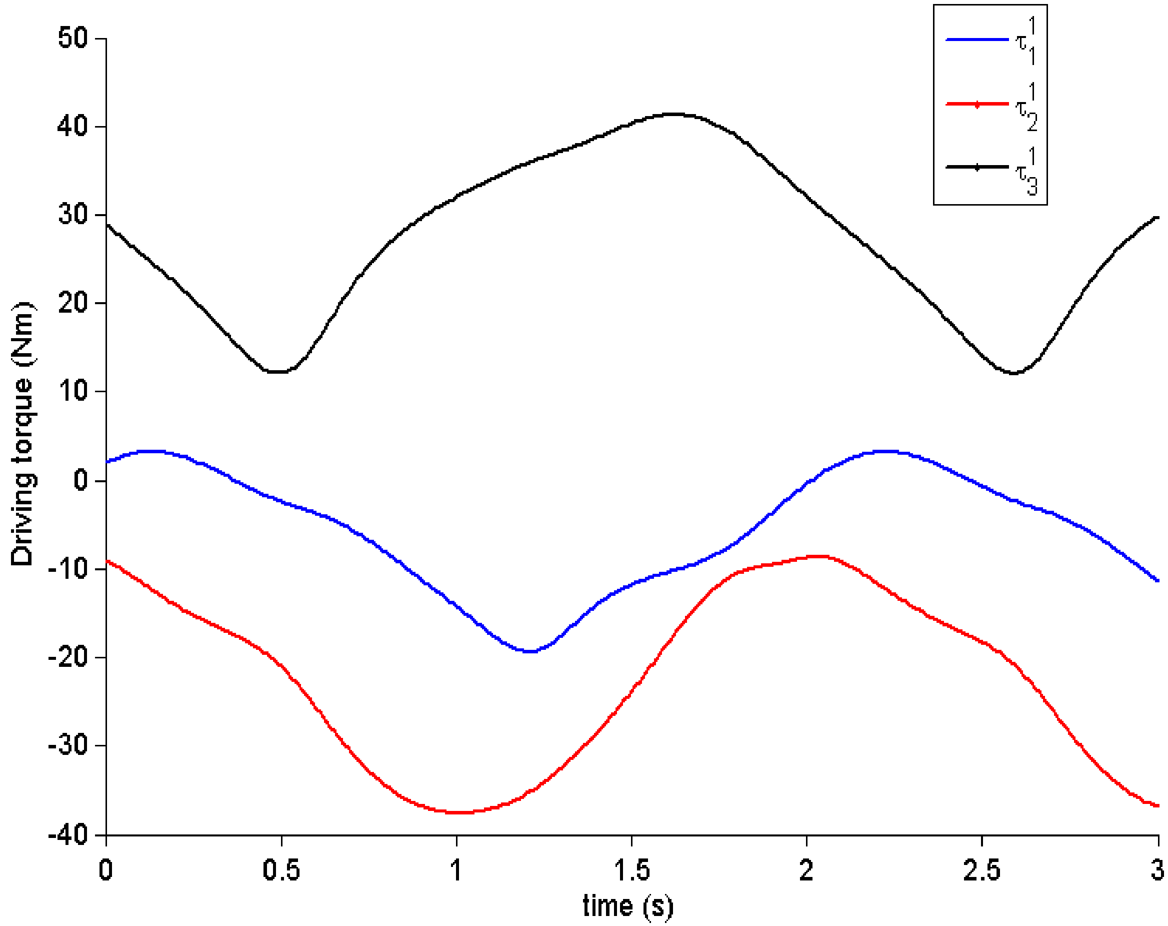
* represents the side of equilateral triangular link.

Inverse and Forward using ReDySim

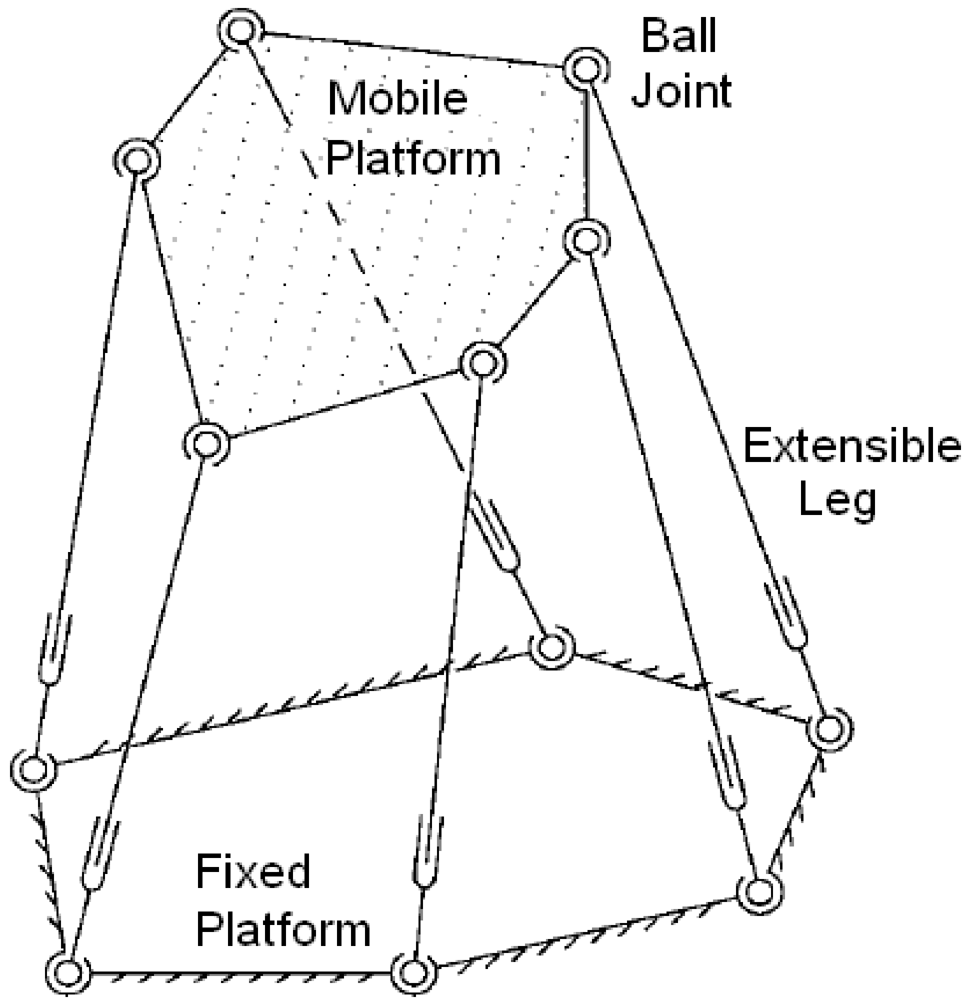
Driving Torques (w/o gravity)

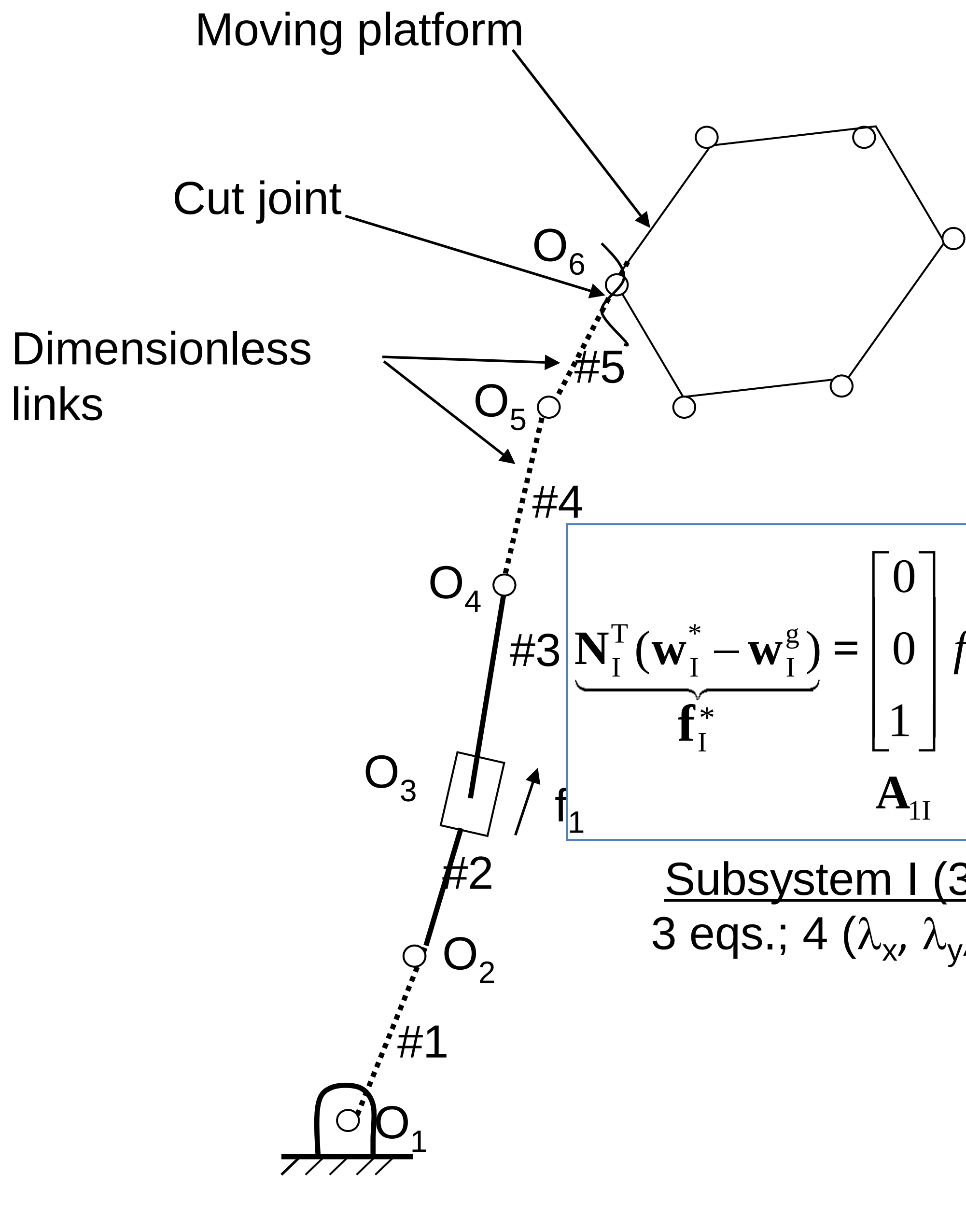


Driving Torques (w/gravity)



Six-DOF Parallel Robot: Stewart Platform



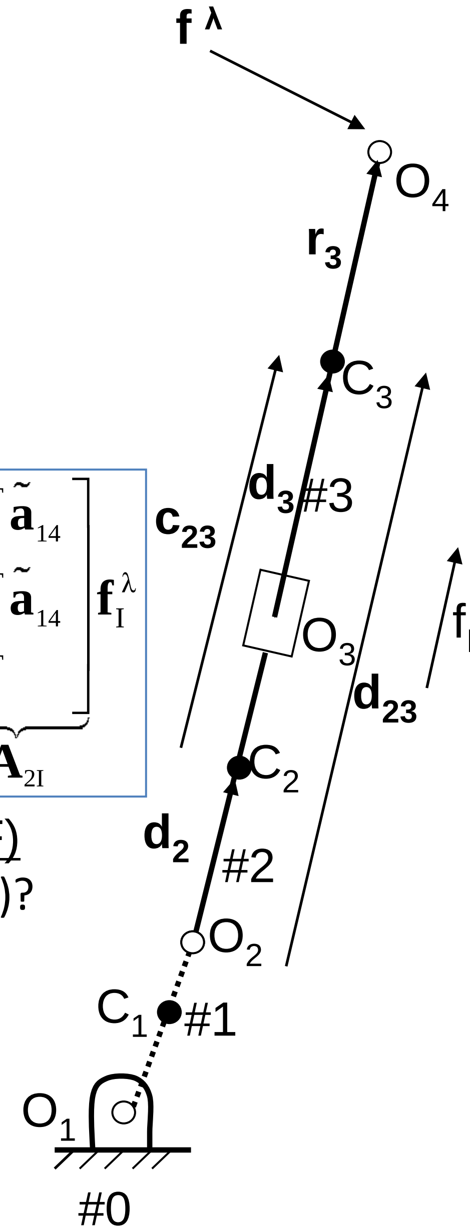


One particular limb

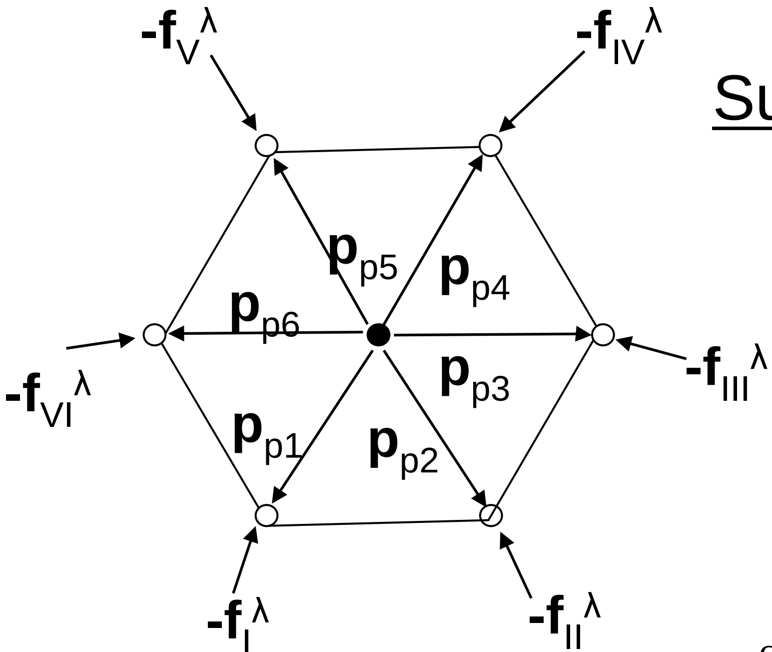
$$\underbrace{\mathbf{N}_I^T(\mathbf{w}_I^* - \mathbf{w}_I^g)}_{\mathbf{f}_I^*} = \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\mathbf{A}_{1I}} f_I + \underbrace{\begin{bmatrix}\mathbf{e}_1^T\tilde{\mathbf{a}}_{14}\\ \mathbf{e}_2^T\tilde{\mathbf{a}}_{14}\\ \mathbf{e}_3^T\end{bmatrix}}_{\mathbf{A}_{2I}} \mathbf{f}_I^\lambda$$

Subsystem I (3-DOF)

3 eqs.; 4 (λ_x, λ_y, λ_x, f_1)?



FBD of VII Subsystem



Subsystem VII (6-DOF)

6 eqs.; Zero?

$$\mathbf{I}^{c}_{VII}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}^{c}_{VII}\,\boldsymbol{\omega} = \mathbf{n}^{c}_{VII}$$

$$m_{VII}\,\dot{\mathbf{c}} = \mathbf{f}^{c}_{VII}$$

$$\mathbf{n}^{c}_{VII} = \left[\tilde{\mathbf{p}}_{p1} \times \mathbf{1} \quad \cdots \quad \tilde{\mathbf{p}}_{p6} \times \mathbf{1}\right] \begin{bmatrix} \mathbf{f}^{\lambda}_{I} \\ \vdots \\ \mathbf{f}^{\lambda}_{VI} \end{bmatrix}$$

$$\mathbf{f}^{c}_{VII} = -\mathbf{f}^{\lambda}_{I} - \cdots - \mathbf{f}^{\lambda}_{VI}$$

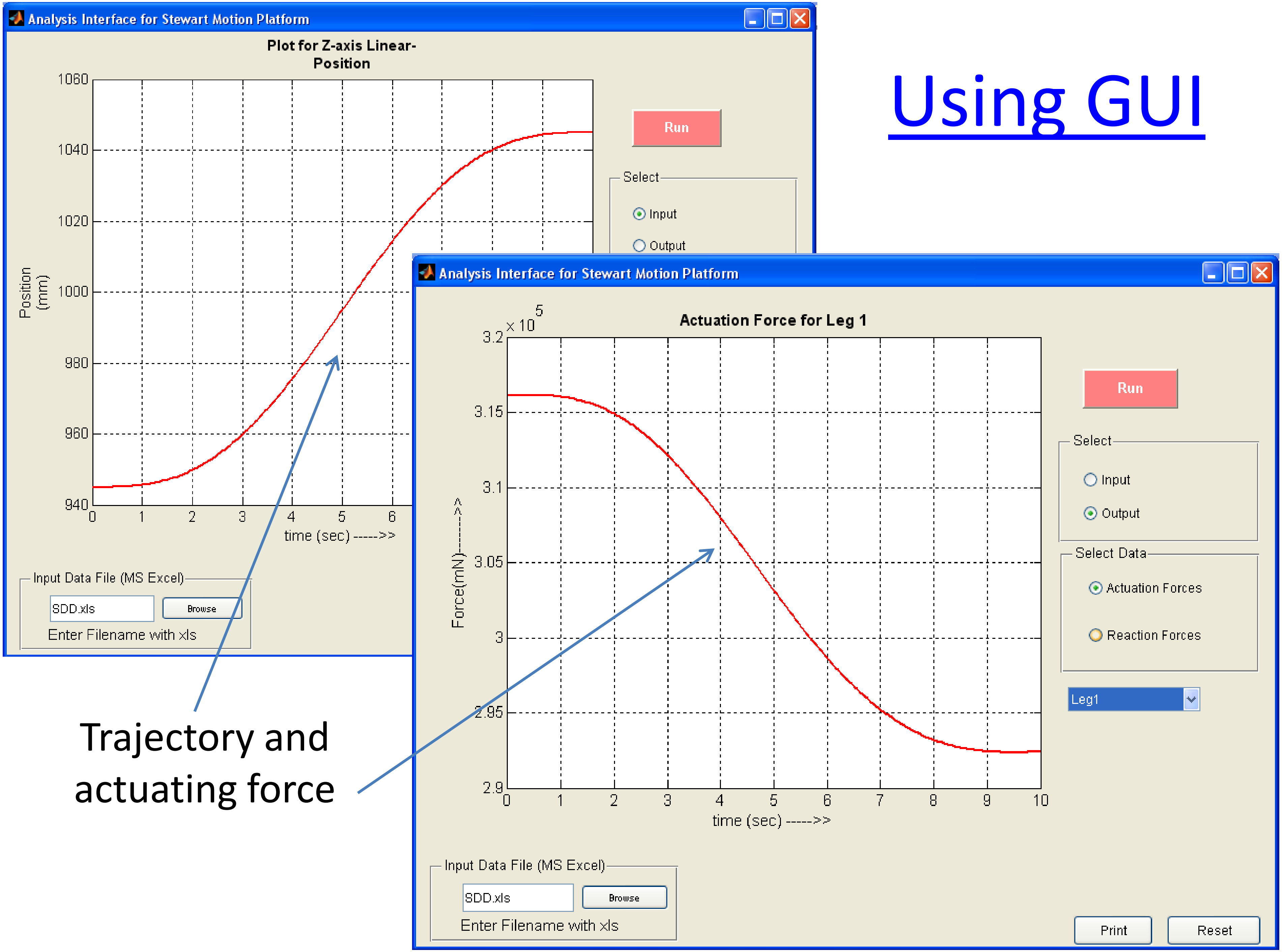
Inverse Dynamics Algorithm

$$\underbrace{\begin{bmatrix} \mathbf{n}^{c}_{\text{VII}} - \sum_{i=1}^{6} \mathbf{x}_i \\ \mathbf{f}^{c}_{\text{VII}} - \sum_{i=1}^{6} \mathbf{y}_i \end{bmatrix}}_{\mathbf{h}:6\times 1} = \underbrace{\begin{bmatrix} \hat{\mathbf{s}}_1 \times \mathbf{p}_{\text{p1}} & \cdots & \hat{\mathbf{s}}_6 \times \mathbf{p}_{\text{p6}} \\ \hat{\mathbf{s}}_1 & \cdots & \hat{\mathbf{s}}_6 \end{bmatrix}}_{\mathbf{J}:6\times 6} \underbrace{\begin{bmatrix} f_I \\ \vdots \\ f_{\text{VI}} \end{bmatrix}}_{\boldsymbol{\tau}:6\times 1}$$

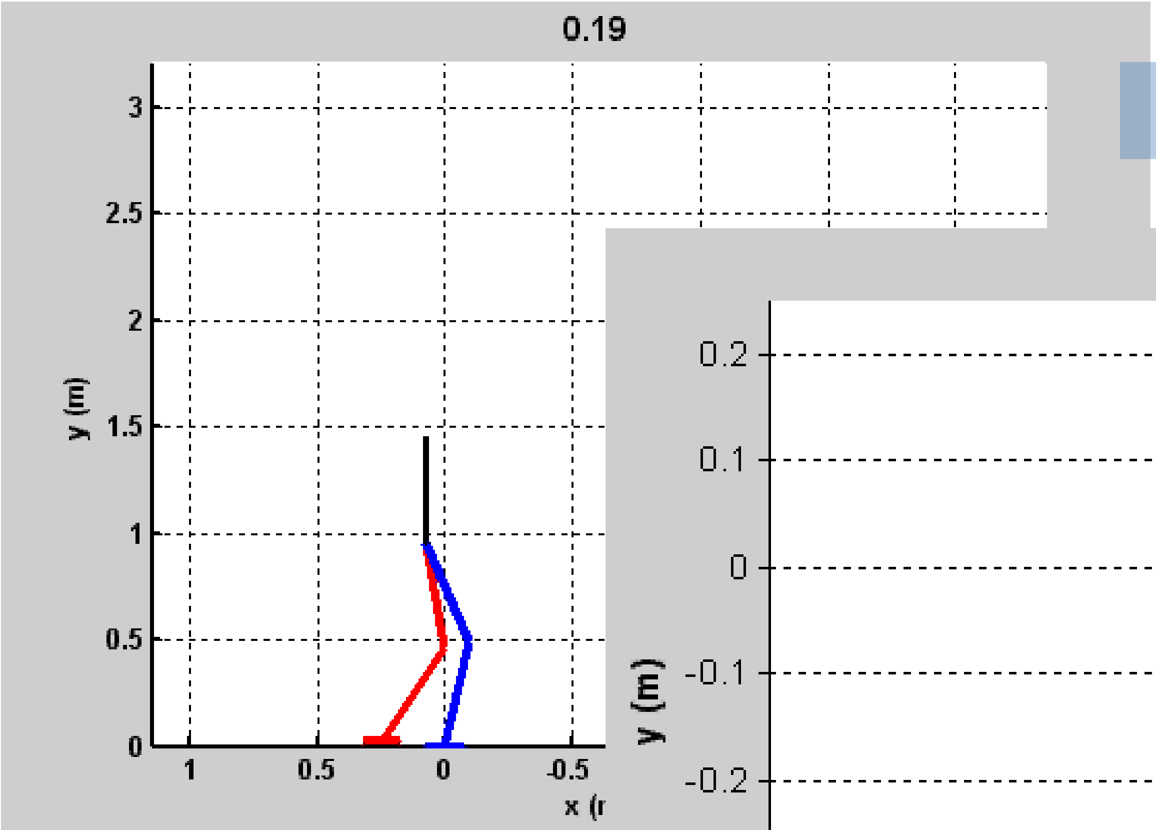
$$\boldsymbol{\tau} = \mathbf{J}^{-1}\mathbf{h}$$

Ref: Sadana, M., 2009, *Dynamic Analysis of 6-DOF Motion Platform*, M. Tech Project Report, IIT Delhi

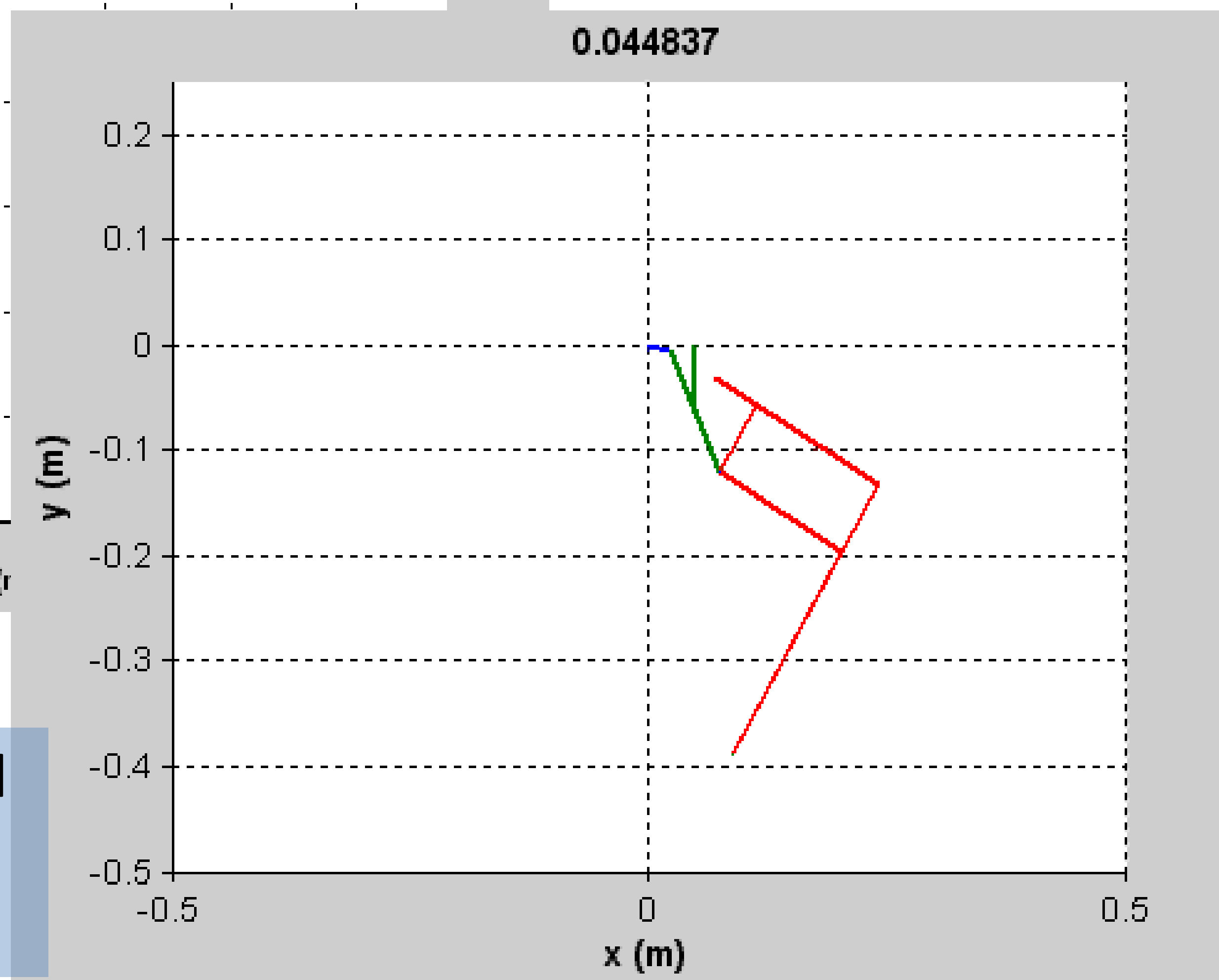
Using GUI


Analysis Interface for Stewart Motion Platform
Plot for Z-axis Linear-Position
Position (mm)
time (sec) ----->>
Run
Select
Input
Output
Input Data File (MS Excel)
SDD.xls
Browse
Enter Filename with xls
Analysis Interface for Stewart Motion Platform
Actuation Force for Leg 1
Force(mN)------>>
time (sec) ----->>
Run
Select
Input
Output
Select Data
Actuation Forces
Reaction Forces
Leg1
Input Data File (MS Excel)
SDD.xls
Browse
Enter Filename with xls
Print
Reset
Trajectory and actuating force

More Robots using ReDySim

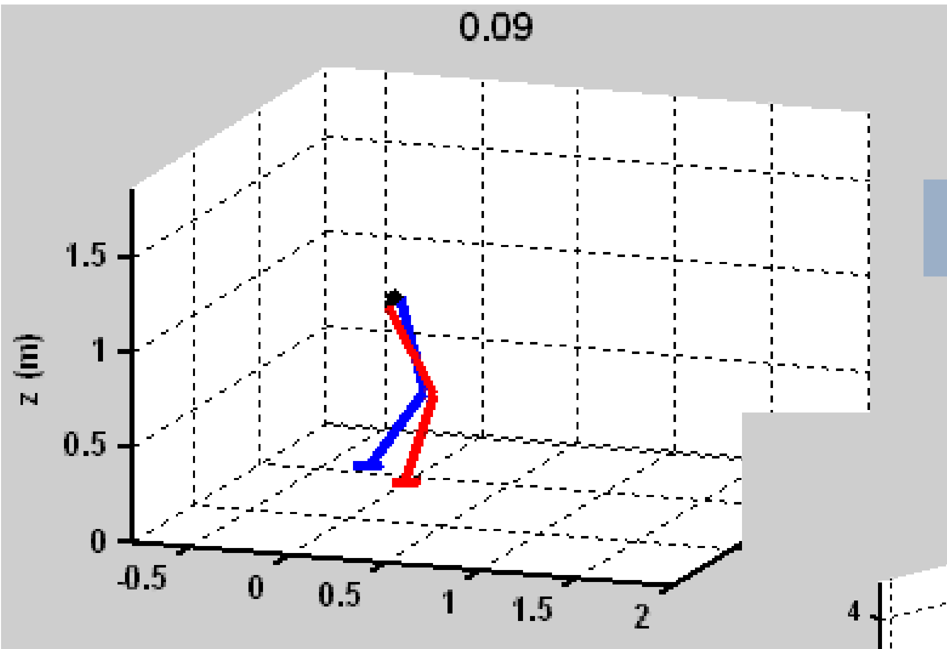


Planar biped

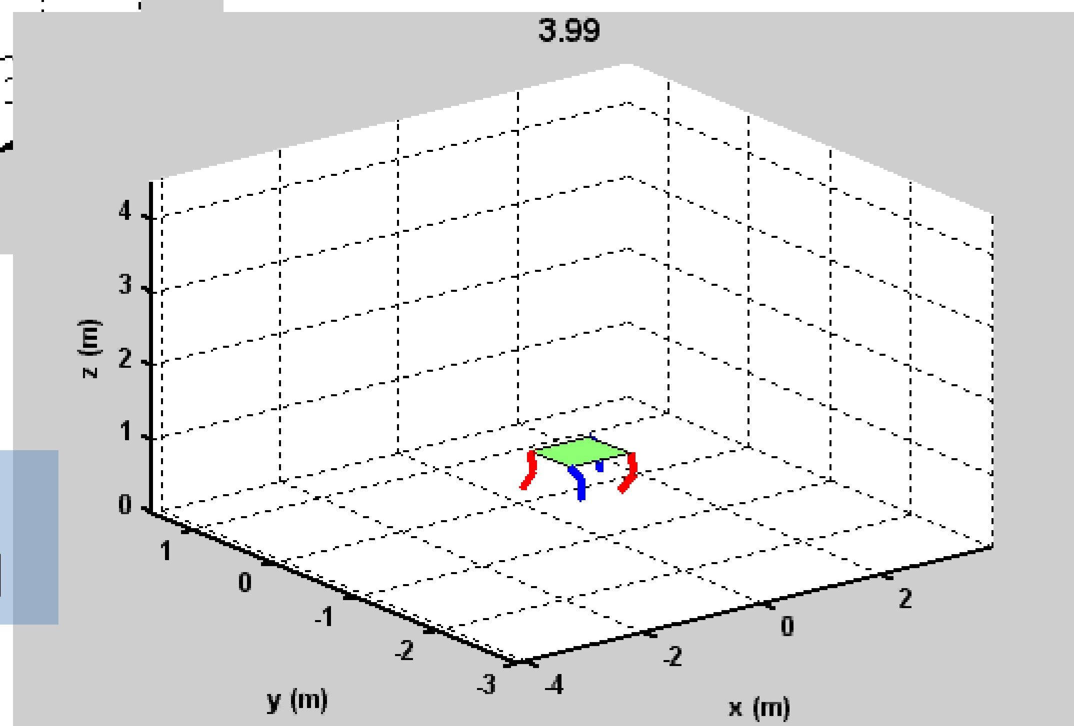


Waking leg (also used as carpet scrapping mechanism)

Legged Robots

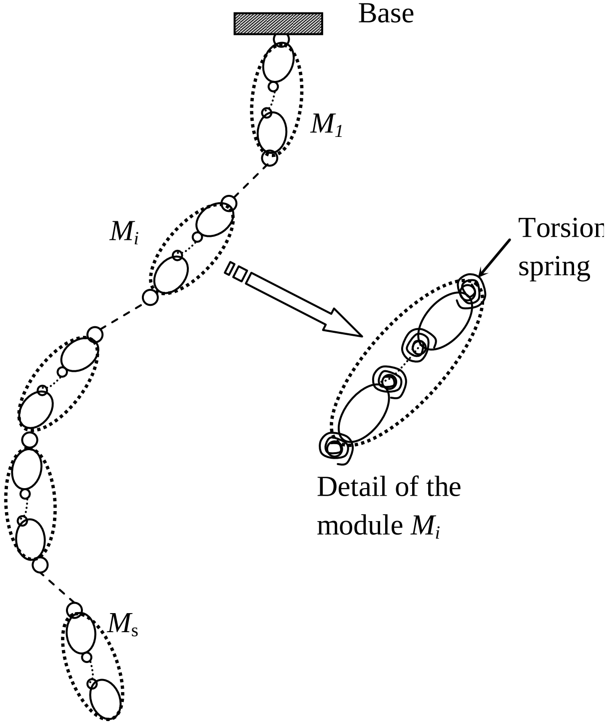


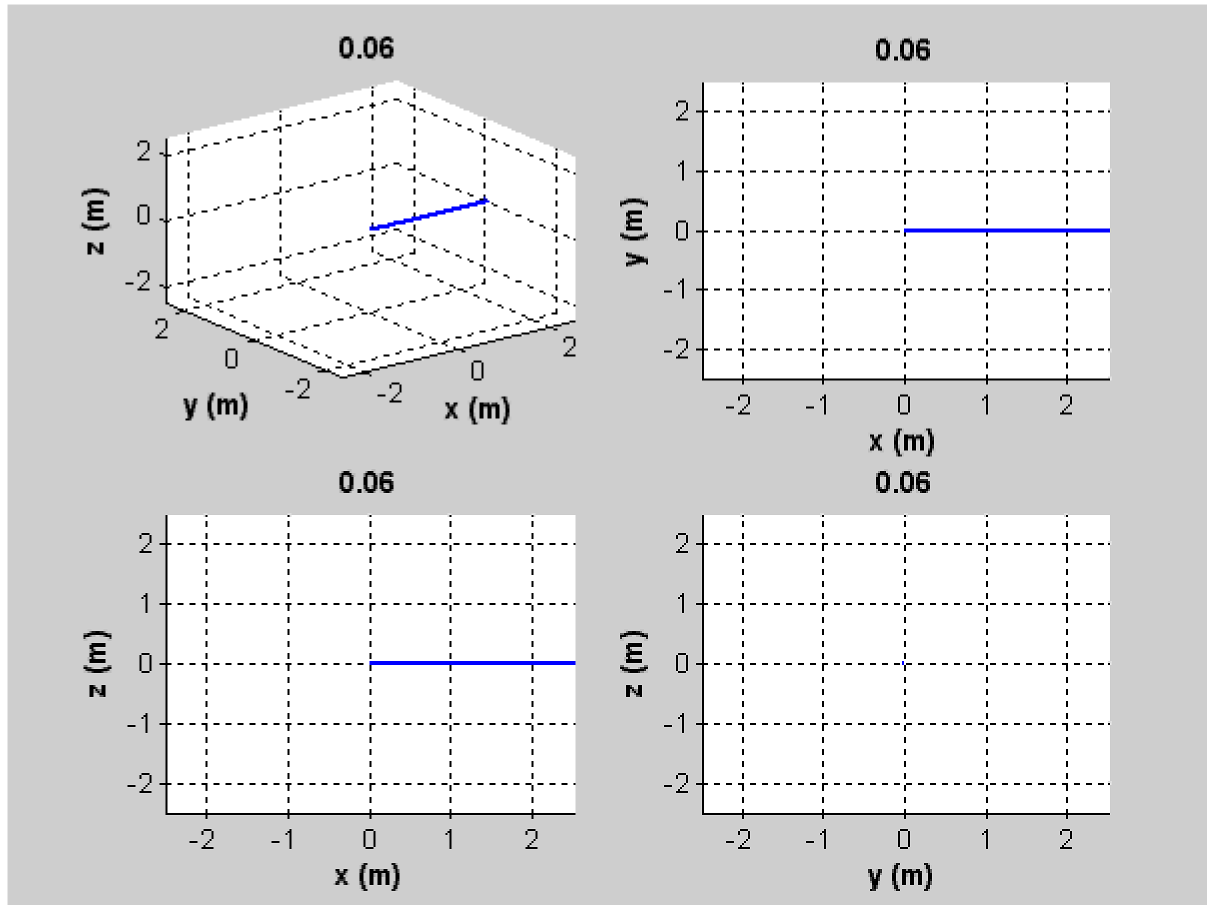
Spatial biped



Spatial quadruped

Flexible Rope using ReDySim





Conclusions

- Purpose of Recursive Robot Dynamics
  - Efficiency
  - Numerical stability
- DeNOC matrices for serial-chain
- NE to EL derivations
- Constrained equations of motion for serial-chain systems
- Schemes for Inverse and Forward Dynamics (Simulation)
- RoboAnalyzer software for robot dynamics
- Parallel robot application
- Four-bar mechanism from FBD
- Cut-open system and subsystem recursion
- ReDySim software
- Five-bar, 3-DOF RRR, Stewart platform dynamics
- Custom-made GUI for Stewart platform
- Simulation of walking robots and rope using ReDySim

Acknowledgements

- Mr. Majid Koul
- Mr. Rajivlochana C.G.
- Dr. Suril V. Shah
- Mr. Mukesh Sadana
- Dr. Himanshu Chaudhary
- Students of IIT Delhi and other institutes who used the materials mentioned in the lectures and given feedbacks.

Thank you

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